Horizontal representation by double Fourier series on the sphere

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1. Motivation

The current operational deterministic model at ECMWF has a horizontal spectral resolution of T511 (gridlength ~40 km), and it is planned to increase the resolution to T799 (gridlength ~25 km) in 2005. In view of the concerns about the efficiency of spectral methods on the sphere as resolutions become higher and higher, it is necessary to keep in mind that a future dynamical core of the ECMWF model might require a complete change in the method of horizontal discretization. One possibility, which would circumvent the increasing cost of the Legendre transforms without requiring a complete change of technique, would be to replace the spherical harmonic basis functions of the conventional spectral technique by double Fourier series. This paper reviews the work which has been done on this topic.

2. Background

The use of double Fourier series on the sphere is motivated not only by the existence of a fast transform algorithm, but also by the close relationship between associated Legendre polynomials and Fourier series. The associated Legendre polynomial with zonal wavenumber m and total wavenumber n can be written (Swarztrauber, 1979) as

$$P_n^m(\sin\theta) = \sum_{k=0}^{n} a_{km} \begin{Bmatrix} \cos(k\phi), & m \quad even \\ \sin(k\phi), & m \quad odd \end{Bmatrix}$$

where θ is latitude and $\phi = \theta + \pi/2$.

Thus, if a function can be represented as a truncated series of associated Legendre functions, it can also be represented by a similarly truncated series of sines or cosines:

$$f_m(\theta) = \sum_{n=m}^{N} \hat{f}_{mn} P_n^m(\sin\theta) = \sum_{k=0}^{N} \tilde{f}_{km} \begin{Bmatrix} \cos(k\phi \\ \sin(k\phi) \end{Bmatrix}$$

Note that the converse is not generally true – for the above equality to hold, the vector of coefficients $\tilde{f}_{km}$ must be restricted to a *subspace* of dimension (*N* + 1-*m*).

The use of double Fourier series leads to accurate horizontal derivatives, fast transforms and easy solution of constant-coefficient elliptic problems. The principal drawback is that pole problems re-emerge unless further steps are taken.

3. History

There is a long history of attempts to construct spectral models on the sphere using double Fourier series. Merilees (1973) coded a *pseudospectral* shallow-water model, but encountered instability (the return of the pole problem). Orszag (1974) formulated a spectral barotropic vorticity model; no results were shown, but he pointed out the expected stability problem. Boyd (1978) successfully used double Fourier series to solve (linear) elliptic and eigenvalue problems on the sphere. Similarly, Yee (1981) demonstrated the use of double Fourier series to solve the Poisson equation on the sphere. Fornberg (1995) and Shen (1999) explored the use of double Fourier series for various (mainly linear) applications on the sphere.

Spotz et al. (1998) based a shallow-water model on double Fourier series, stabilizing the computation by using a "spherical harmonic projection". In terms of the above discussion, this projects the vector $\tilde{f}_{km}$ into the appropriate subspace, so that the model becomes algebraically equivalent to one based on spherical harmonics. The computational complexity of the projection is $O(N^3)$ per timestep, the same as for the Legendre transforms, though the operation count is somewhat reduced. Cheong (2000a) pursued a similar idea ("spherical harmonic filtering") for the barotropic vorticity equation.

Finally, Cheong (2000b) formulated a double Fourier series shallow-water model, but had to include some filtering to prevent instability. Layton and Spotz (2003) built such a model including semi-Lagrangian advection, but still found that a spherical harmonic projection was needed to stabilize the results.

4. Choice of basis functions

The relationships between associated Legendre polynomials and Fourier series suggest that for odd zonal wavenumber m, an expansion in terms of $\sin k\phi$ is appropriate. It is easily seen that if a variable (for odd m) is expanded in such a series, then both X and $\partial X / \partial \phi$ behave correctly (continuously) at the poles. For zonal wavenumber $m = 0$, it is similarly clear that an expansion in terms of $\cos k\phi$ is appropriate. For even zonal wavenumber $m > 0$, Orszag (1974) and most subsequent authors have again chosen an expansion in terms of $\cos k\phi$; then $\partial X / \partial \phi$ behaves correctly at the poles, but additional constraints have to be imposed in order to force X itself to be zero at the poles. Cheong (2000b) has suggested an alternative expansion in terms of $\sin\phi \sin k\phi$ for the case m even, $m>0$ since both X and $\partial X / \partial \phi$ then behave correctly at the poles.

In the case of expansions in terms of spherical harmonics, it is natural to choose a "triangular" truncation in spectral space, since this is equivalent to an isotropic resolution on the sphere. In the case of double Fourier series expansions, the appropriate "shape" of the truncation in spectral space (rectangular? elliptic?) will have to be determined by experiment.

5. Example: Poisson equation

Since the spherical harmonics are eigenfunctions of the Laplacian operator on the sphere, expansions in terms of these functions convert the Poisson equation into a simple diagonal problem in spectral space. Yee (1981) and, using slightly different basis functions, Cheong (2000a) have demonstrated the use of double Fourier series expansions to solve the Poisson equation over the sphere. In this case we obtain, for each zonal wavenumber m, a pair of tridiagonal systems to solve in spectral space – one for odd values of the meridional index k, and one for the even values. Similar considerations apply for the Helmholtz equation which typically results from the use of a semi-implicit time integration scheme.

6. The pole problem

For linear problems such as the elliptic equations described above, issues such as the aliasing of nonlinear terms or instabilities appearing during time integration do not arise, and the use of double Fourier series is relatively straightforward. Things may not be so simple when we move to nonlinear time-dependent problems on the sphere. In particular, difficulties are likely to be encountered in the vicinity of the poles.

For efficiency, in our current spectral model based on spherical harmonics, we use a reduced grid (Δx ~constant) to give approximately uniform resolution over the sphere (Hortal and Simmons, 1991). There is some hope that the reduced grid would be sufficient to control the pole problem when using double Fourier series, since such a grid cannot support high zonal wavenumbers near the pole – but this remains to be demonstrated in practice.

7. Context and outlook

We have nearly completed the coding to test a double Fourier series formulation of the shallow water equations on the sphere, including the Williamson et al. (1992) test problems. Some options (e.g., the precise choice of basis functions and the shape of the spectral truncation) have been left open for now and will be the subject of experimentation. In the meantime, we can take comfort in the fact that at a horizontal resolution of T799 (gridlength ~25km) with 91 levels, the proportion of the computation time spent in the Legendre transforms is only about 7% of the total timestep **-** so the "Legendre barrier" to further increases in horizontal resolution is still some way distant.

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