Atmospheric Waves

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Based on lectures by M. Miller and J.S.A. Green

Introduction

Why study atmospheric waves in a course on numerical modelling?

Successful numerical modelling requires a good understanding of the system under investigation and its solutions.

Waves are important solutions of the atmospheric system.

Waves can be numerically demanding!

(E.g. acoustic waves
have high frequencies and large phase speeds
=> short time step in numerical integrations with
explicit time-stepping schemes.)

Introduction (2)

If a wave type is not of interest and a numerical nuisance then filter these waves out, i.e. modify the governing equations such that this wave type is suppressed.

How?

To know how, we need to have a good understanding of the wave solutions and of which terms in the equations are responsible for the generation of the individual wave types.

Question: How do the modifications to the governing equations to eliminate unwanted wave types affect the wave types we want to retain and study?

Also: How do other commonly made approximations (e.g. hydrostatic approximation) affect the wave solutions?

Introduction (3)

How are we going to address these questions?

Ideally, we have to study analytically the wave solutions of the exact set of governing equations for the atmosphere first.

Then we introduce approximations and study their effect on the wave solutions by comparing the new solutions with the exact solutions.

Problem with this approach:

Governing equations of the atmosphere are non-linear (e.g. advection terms) and cannot be solved analytically in general!

We linearize the equations and study here the linear wave solutions analytically.

Introduction (4)

Question

Are these linear wave solutions representative of the non-linear solutions?

Answer

Yes, to some degree.

Non-linearity can considerably modify the linear solutions but does not introduce new wave types!

Therefore, the origin of the different wave types can be identified in the linearized system and useful methods of filtering individual wave types can be determined and then adapted for the non-linear system.

Objectives of this course

- Discuss the different wave types which can be present in the atmosphere and the origin of these wave types.
- Derive filtering approximations to filter out or isolate specific wave types.
- Examine the effect of these filtering approximations and other commonly made approximations on the different wave types present in the atmosphere.

Method:

Find analytically the wave solutions of the linearized basic equations of the atmosphere, first without approximations.

Introduce approximations later and compare new solutions with the exact solutions.

Definition of basic wave properties (1)

Mathematical expression for a 2-dimensional harmonic wave

$$\boxed{\Psi(x,z,t) = A \cdot \exp\left[i\left(kx + mz - \sigma t\right)\right]}$$

Amplitude A

Wave numbers $\boxed{k = \frac{2\pi}{L_x}, \quad m = \frac{2\pi}{L_z}}$ Wave lengths $\boxed{L_x, L_z}$

Wave vector $\boxed{\vec{K} \equiv (k, m)}$

Frequency $\boxed{\sigma = \frac{2\pi}{T}}$ Period T

Dispersion relation $\boxed{\sigma(k, m, \text{parameters of the system})}$

Definition of basic wave properties (2)

$$\boxed{\Psi(x,z,t) = A \cdot \exp\left[i\left(kx + mz - \sigma t\right)\right]}$$

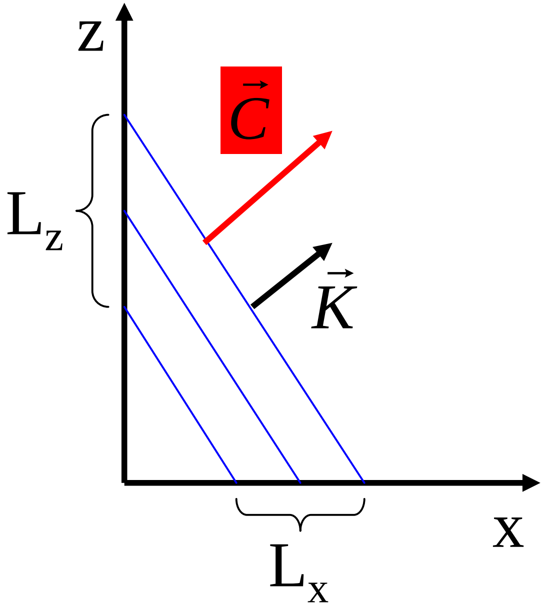
Phase: $\boxed{\varphi \equiv kx + mz - \sigma t = \vec{K}\cdot\vec{X} - \sigma t}$ where $\vec{X} = (x, z)$

Wave fronts or phase lines

= lines of constant phase

(that is, all $\vec{X}$ for which $\vec{K}\cdot\vec{X} - \sigma t = const.$)

$\vec{K}$ is perpendicular to the wave fronts.



Phase velocity $\vec{C}$ = velocity of wave fronts.

$$\frac{D\varphi}{Dt} = \vec{K}\cdot\underbrace{\frac{D\vec{X}}{Dt}}_{=\vec{C}} - \sigma = 0 \quad =>$$

$$\boxed{\vec{C} = \left(\frac{\sigma k}{k^2 + m^2}\ ,\ \frac{\sigma m}{k^2 + m^2}\right)}$$

Definition of basic wave properties (3)

Horizontal phase velocity $$c_x \equiv \frac{\sigma}{k}$$

Vertical phase velocity $$c_z \equiv \frac{\sigma}{m}$$

!!! $(c_x , c_z) \underline{\neq \vec{C}}$!!!

Dispersive waves are waves with a phase velocity that depends on the wave number.

Wave packet is a superposition of individual waves.

Group velocity $$\boxed{\vec{C}_g \equiv \left(\frac{\partial\sigma}{\partial k} , \frac{\partial\sigma}{\partial m} \right)}$$

Energy is transmitted with the group velocity.

Waves travel with the phase velocity.

Basic Equations

We use height (z) as vertical coordinate.

Momentum equations:

$$\begin{cases} \dfrac{Du}{Dt} - f\mathrm{v} = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial x} & (1) \\[2ex] \dfrac{D\mathrm{v}}{Dt} + fu = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial y} & (2) \\[2ex] \dfrac{Dw}{Dt} + g = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial z} & (3) \end{cases}$$

Continuity equation:

$$\frac{\partial u}{\partial x} + \frac{\partial \mathrm{v}}{\partial y} + \frac{\partial w}{\partial z} = \frac{D(\ln \rho)}{Dt} \qquad (4)$$

Thermodynamic equation:

$$\frac{D(\ln T)}{Dt} = \kappa \frac{D(\ln p)}{Dt} \qquad (5)$$

Complemented by Equation of state: $p = \rho RT$

Remarks:

(1) All source/sink terms are omitted in eqs (1)-(5)

(2) Total time derivative is defined as

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + \mathrm{v}\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$$

(3) $\kappa \equiv \dfrac{R}{c_p}$, where R is the ideal gas constant and c_p the heat capacity at constant pressure

(4) Setting $\dfrac{Dw}{Dt} \equiv 0$ gives familiar set of equations in hydrostatic approximation

We don't make the hydrostatic approximation at present!

It will be discussed later in detail.

We would like to find analytically the wave solutions for the basic equations (1)-(5).

Can we do this?

No! Basic equations are non-linear partial differential equations!

We have to linearize the basic equations (1)-(5) by using the perturbation method and solve the linearized system analytically.

Introduce first some simplifications:

Change of variable: Replace T by $\boxed{\Theta \equiv \ln\theta}$ where

$$\boxed{\theta \equiv T\left(\frac{p_0}{p}\right)^{\kappa}}$$

(potential temperature)

Simplification: Neglect variation in y $\boxed{\frac{\partial}{\partial y} \equiv 0}$ $(e.g. \frac{\partial f}{\partial y} = 0\ )$

Coordinates are now only (x, z, t)!

Question: Has this simplification serious consequences for the wave solutions?

Answer: Yes! The Rossby wave solution has been suppressed!! Rossby waves can only form if the Coriolis parameter f changes with latitude. (Detailed discussion of Rossby waves will follow later in this course.)

Dependent variables (unknowns) are now u, v, w, ρ, p, Θ and they are functions only of x, z and t.

Now we linearize the set of equations (1)-(5) by using the perturbation method.

Perturbation Method

All field variables are divided into 2 parts:
1) a basic state part
2) a perturbation part (= local deviation from the basic state)

$$u = \boxed{u_0} + \boxed{\delta u}$$

Basic assumptions of perturbation theory are:
a.) The basic state variables must themselves satisfy the governing equations.
b.) Perturbations must be small enough to neglect all products of perturbations.

=> Non-linear equations are reduced to linear differential equations in the perturbation variables in which the basic state variables are specified coefficients.

Apply perturbation method to basic equations (1)-(5)

Consider small perturbations on an initially motionless atmosphere, i.e. basic state winds (u_0,v_0,w_0)=0

$$u = u_0 + \delta u = \delta u(x,z,t)$$

$$\mathrm{v} = \mathrm{v}_0 + \delta \mathrm{v} = \delta \mathrm{v}(x,z,t)$$

$$w = w_0 + \delta w = \delta w(x,z,t)$$

$$\rho = \rho_0(z) + \delta\rho(x,z,t)$$

$$p = p_0(z) + \delta p(x,z,t)$$

$$\Theta = \Theta_0(z) + \delta\Theta(x,z,t)$$

ρ_0, p_0, Θ_0 define the basic atmospheric state and satisfy $\boxed{\dfrac{\partial p_0}{\partial z} = -g\rho_0.}$

Inserting into (1)-(5) and neglecting products of perturbations gives linearized basic equations.

Linearized basic equations

Perturbations (δu, δv, etc.) are now the dependent variables!

$$
\begin{aligned}
&\frac{\partial \delta u}{\partial t} - f\delta \mathrm{v} + \frac{\partial}{\partial x}\left(\frac{\delta p}{\rho_0}\right) &&= 0\\
&\frac{\partial \delta \mathrm{v}}{\partial t} + f\delta u &&= 0\\
&\frac{\partial \delta w}{\partial t} + \frac{\partial}{\partial z}\left(\frac{\delta p}{\rho_0}\right) - \underline{B}\,\frac{\delta p}{\rho_0} - g\,\delta\Theta &&= 0\\
&\frac{\partial}{\partial t}\left(\frac{\delta \rho}{\rho_0}\right) + \frac{\partial \delta u}{\partial x} + \frac{\partial \delta w}{\partial z} - \frac{\delta w}{\underline{H_0}} &&= 0\\
&\frac{\partial \delta\Theta}{\partial t} + \underline{B}\,\delta w &&= 0
\end{aligned}
$$

No advection terms left!

Here:

$$B \equiv \frac{\partial}{\partial z}(\ln \theta_0)$$

static stability

$$\frac{1}{H_0} \equiv -\frac{\partial}{\partial z}(\ln \rho_0)$$

density scale height

For this set of equations it is now possible to find the wave solutions analytically.

Introduction of tracer parameters

Trick to help us save work and make sensible approximations later.

Introduce tracer parameters n_1, n_2, n_3 and n_4 to “mark” individual terms in the equations who’s effect on the solutions we want to investigate.

These tracers have the value 1 but may individually be set to 0 to eliminate the corresponding term.

For example $n_4 = 0$ =>
hydrostatic approx. to pressure field.

$$\frac{\partial \delta u}{\partial t} - f\delta \mathrm{v} + \frac{\partial}{\partial x}\left(\frac{\delta p}{\rho_0}\right) = 0 \qquad (17)$$

$$\frac{\partial \delta \mathrm{v}}{\partial t} + f\delta u = 0 \qquad (18)$$

$$\enclose{circle}{n_4}\frac{\partial \delta w}{\partial t} + \frac{\partial}{\partial z}\left(\frac{\delta p}{\rho_0}\right) - \enclose{circle}{n_3} B\frac{\delta p}{\rho_0} - g\delta\Theta = 0 \qquad (19)$$

$$\enclose{circle}{n_2}\frac{\partial}{\partial t}\left(\frac{\delta \rho}{\rho_0}\right) + \frac{\partial \delta u}{\partial x} + \frac{\partial \delta w}{\partial z} - \enclose{circle}{n_1}\frac{\delta w}{H_0} = 0 \qquad (20)$$

$$\frac{\partial \delta\Theta}{\partial t} + B\delta w = 0 \qquad (21)$$

Find wave solutions for system of linearized equations (17)-(21):

Boundary conditions:

For simplicity we assume the atmosphere to be unbounded in x and z.

Wave solutions:

Since the coefficients f, B, g & H_0 of the system (17)-(21) are independent of x & t, the solutions can be written in the from

$$F(z)\exp\{i(kx+\sigma t)\}\ .$$

Each dependent variable (perturbation) is of this form:

$$\delta u = \hat{u}(z)\cdot\exp\{i(kx+\sigma t)\}$$
$$\delta \mathrm{v} = \hat{\mathrm{v}}(z)\cdot\exp\{i(kx+\sigma t)\}$$
$$\ldots$$

Remarks:

a.) The full solution is the appropriate Fourier sum of terms of this form over all wave numbers k. We study here only individual waves.

b.) If the frequency is complex we have amplifying or decaying waves in time. We study only "neutral" waves, so σ is assumed to be real.

Inserting $\delta u = \hat{u}(z)\cdot\exp\{i(kx+\sigma t)\}, \delta \mathrm{v} = \hat{\mathrm{v}}(z)\cdot\exp\{i(kx+\sigma t)\}$ etc. into eqs (17)-(21) gives the following set of ordinary differential equations in *z* (derivatives only in *z*!):

$$\underline{i\sigma}\hat{u} - f\hat{\mathrm{v}} + \underline{ik}\frac{\hat{p}}{\rho_0} = 0 \qquad (22)$$

$$\underline{i\sigma}\hat{\mathrm{v}} + f\hat{u} = 0 \qquad (23)$$

$$n_4\underline{i\sigma}\hat{w} + \boxed{\frac{d}{dz}}\left(\frac{\hat{p}}{\rho_0}\right) - n_3 B\frac{\hat{p}}{\rho_0} - g\hat{\Theta} = 0 \qquad (24)$$

$$n_2\underline{i\sigma}\frac{\hat{\rho}}{\rho_0} + \underline{ik}\hat{u} + \boxed{\frac{d}{dz}}\hat{w} - \frac{n_1}{H_0}\hat{w} = 0 \qquad (25)$$

$$\underline{i\sigma}\hat{\Theta} + B\hat{w} = 0 \qquad (26)$$

No *x* and *t* dependencies left! Operators $\partial/\partial x$ and $\partial/\partial t$ have been replaced by ik and $i\sigma$, respectively.

Dependent variables are now $\hat{u}(z), \hat{\mathrm{v}}(z), \hat{w}(z), \hat{p}(z), \hat{\rho}(z)$, and $\hat{\Theta}(z)$.

Solve system of equations (22)-(26) for

$\hat{u}(z)$, $\hat{v}(z)$, $\hat{w}(z)$, $\hat{p}(z)$, $\hat{\rho}(z)$, and $\hat{\Theta}(z)$:

Strategy:

Step 1:

Derive from this set of equations one differential equation in only one of the dependent variables: $\hat{w}(z)$.

Step 2:

Find solution of this equation, i.e. $\hat{w}(z)$ and the dispersion relationship $\sigma(k, m, \textit{parameters of the system})$.

Step 3:

Insert this solution for $\hat{w}(z)$ back into (22)-(26) to obtain solutions for the remaining dependent variables.

Deriving from (22)-(26) a differential equation only in $\hat{w}(z)$ 1

$$i\sigma\hat{u} - f\hat{v} + ik\frac{\hat{p}}{\rho_0} = 0 \quad (22)$$

$$i\sigma\hat{v} + f\hat{u} = 0 \quad (23)$$

$$n_4 i\sigma\hat{w} + \frac{d}{dz}\left(\frac{\hat{p}}{\rho_0}\right) - n_3 B\frac{\hat{p}}{\rho_0} - g\hat{\Theta} = 0 \quad (24)$$

$$n_2 i\sigma\frac{\hat{\rho}}{\rho_0} + ik\hat{u} + \frac{d}{dz}\hat{w} - \frac{n_1}{H_0}\hat{w} = 0 \quad (25)$$

$$i\sigma\hat{\Theta} + B\hat{w} = 0 \quad (26)$$

From (22) and (23) we obtain

$$\hat{u} = -\frac{\sigma k}{\sigma^2 - f^2}\frac{\hat{p}}{\rho_0} \quad (27)$$

$$\hat{v} = -\frac{ifk}{\sigma^2 - f^2}\frac{\hat{p}}{\rho_0} \quad (28)$$

Inserting $\hat{u}$ from (27) into (25), using (26) and the relation

$$\hat{\Theta} = \frac{1}{\gamma}\frac{\hat{p}}{p_0} - \frac{\hat{\rho}}{\rho_0} = \frac{1}{c^2}\frac{\hat{p}}{\rho_0} - \frac{\hat{\rho}}{\rho_0}$$

, where $c \equiv \sqrt{\gamma R T_0}$ is the Laplacian speed of sound,

transforms (25) into

$$\frac{d}{dz}\hat{w} + \left(Bn_2 - \frac{n_1}{H_0}\right)\hat{w} + i\sigma\left(\frac{n_2}{c^2} - \frac{k^2}{\sigma^2 - f^2}\right)\frac{\hat{p}}{\rho_0} = 0 \quad (29)$$

Using (26) to eliminate $\hat{\Theta}$ from (24) gives

$$i\sigma\frac{d}{dz}\left(\frac{\hat{p}}{\rho_0}\right) - i\sigma B n_3\frac{\hat{p}}{\rho_0} + (gB - n_4\sigma^2)\hat{w} = 0 \quad (31)$$

Deriving from (22)-(26) a differential equation only in $\hat{w}(z)$ 2

$$\left\{\begin{aligned}
&\underline{\frac{d}{dz}}\hat{w}+\left(Bn_2-\frac{n_1}{H_0}\right)\underline{\hat{w}}+i\sigma\left(\frac{n_2}{c^2}-\frac{k^2}{\sigma^2-f^2}\right)\frac{\hat{p}}{\underline{\rho_0}}=0 \quad (29)\\
&i\sigma\,\underline{\frac{d}{dz}\left(\frac{\hat{p}}{\rho_0}\right)}-i\sigma Bn_3\frac{\hat{p}}{\underline{\rho_0}}+(gB-n_4\sigma^2)\underline{\hat{w}}=0 \quad (31)
\end{aligned}\right.$$

In general the coefficients B, H_0 and c are (known) functions of z. For simplicity we consider only constant (mean) values of B, H_0 and c which are related by $B+g/c^2=1/H_0$.

Computing $\hat{p}/\rho_0$ from (29) and inserting into (31) leads to the following second order ordinary differential equation governing the height variation of $\hat{w}$

$$\sigma\left\{\underline{\frac{d^2}{dz^2}}+\left[B(n_2-n_3)-\frac{n_1}{H_0}\right]\underline{\frac{d}{dz}}+(gB-n_4\sigma^2)\left(\frac{k^2}{\sigma^2-f^2}-\frac{n_2}{c^2}\right)-Bn_3\left(Bn_2-\frac{n_1}{H_0}\right)\right\}\underline{\hat{w}(z)}=0 \quad (32)$$

Finished step 1 !!!!

Exercise:

Verify that the following relation between the constants of the system holds exactly in a hydrostatic pressure field.

$$\boxed{B + g/c^2 = 1/H_0}$$

You need the following definitions

$$B \equiv \frac{d}{dz}(\ln\theta), \quad \theta \equiv T\left(\frac{p_0}{p}\right)^{\kappa}, \quad \kappa \equiv \frac{R}{c_p}, \quad c^2 \equiv \gamma RT, \quad \gamma \equiv \frac{c_p}{c_v}, \quad \frac{1}{H_0} \equiv -\frac{d}{dz}(\ln\rho)$$

plus the gas law $p = \rho RT$, the hydrostatic equation $\frac{dp}{dz} = -g\rho$

and to know that the gas constant R is related to the specific heat constants as

$$R = c_p - c_v\,.$$

Solutions of equation (32)

$$\sigma\left\{\frac{d^2}{dz^2}+\left[B(n_2-n_3)-\frac{n_1}{H_0}\right]\frac{d}{dz}+(gB-n_4\sigma^2)\left(\frac{k^2}{\sigma^2-f^2}-\frac{n_2}{c^2}\right)-Bn_3\left(Bn_2-\frac{n_1}{H_0}\right)\right\}\hat{w}(z)=0 \qquad (32)$$

Solution 1: $\boxed{\sigma=0}$ Not a wave!

Inserting $\sigma = 0$ into (22)-(26) gives for the winds:

$\hat{u}=\hat{w}=0$ and $\hat{v}=\frac{ik}{f}\frac{\hat{p}}{\rho_0} \Rightarrow \delta v=\frac{1}{f\rho_0}\frac{\partial}{\partial x}\delta p$, i.e. geostrophic motion.

Solution 2: $\boxed{\hat{w}\equiv 0 \ \forall z}$ Lamb wave

This solution will be discussed later in detail.

Further solutions: For $\sigma\neq 0$ and $\hat{w}\neq 0\ \forall z$ we have to solve

$$\frac{d^2\hat{w}}{dz^2}+\left[B(n_2-n_3)-\frac{n_1}{H_0}\right]\frac{d\hat{w}}{dz}+\left[(gB-n_4\sigma^2)\left(\frac{k^2}{\sigma^2-f^2}-\frac{n_2}{c^2}\right)-Bn_3\left(Bn_2-\frac{n_1}{H_0}\right)\right]\hat{w}=0 \qquad (32a)$$

Finding wave solutions of equation (32a)

$$\frac{d^2\hat{w}}{dz^2} + \left[B(n_2 - n_3) - \frac{n_1}{H_0}\right]\frac{d\hat{w}}{dz} + \left[(gB - n_4\sigma^2)\left(\frac{k^2}{\sigma^2 - f^2} - \frac{n_2}{c^2}\right) - Bn_3\left(Bn_2 - \frac{n_1}{H_0}\right)\right]\hat{w} = 0 \tag{32a}$$

Setting $\hat{w}(z) = \tilde{w}(z)\exp\left\{-\frac{1}{2}\left[B(n_2 - n_3) - \frac{n_1}{H_0}\right]z\right\}$ leads to a simpler differential equation for $\tilde{w}(z)$ with <u>no first derivatives</u>

$$\frac{d^2\tilde{w}}{dz^2} + \left\{(gB - n_4\sigma^2)\left(\frac{k^2}{\sigma^2 - f^2} - \frac{n_2}{c^2}\right) - Bn_3\left(Bn_2 - \frac{n_1}{H_0}\right) - \frac{1}{4}\left[B(n_2 - n_3) - \frac{n_1}{H_0}\right]^2\right\}\tilde{w} = 0 \tag{32b}$$

(32b) has the form of a wave equation and, since we consider the fluid to be <u>unbounded</u> in *z*, $\tilde{w} \propto \exp(imz)$ is solution if *m* fulfills

$$\underline{m^2} = (gB - n_4\sigma^2)\left(\frac{\underline{k^2}}{\sigma^2 - f^2} - \frac{n_2}{c^2}\right) - Bn_3\left(Bn_2 - \frac{n_1}{H_0}\right) - \frac{1}{4}\left[B(n_2 - n_3) - \frac{n_1}{H_0}\right]^2 \tag{33}$$

Dispersion relationship (of 4th order in the frequency *σ)*

Final form of the solution for the perturbation δw

From $\delta w(x,z,t) = \hat{w}(z)\cdot\exp\{i(kx+\sigma t)\}$

with $\hat{w}(z) = \tilde{w}(z)\exp\left\{-\frac{1}{2}\left[B(n_2-n_3)-\frac{n_1}{H_0}\right]z\right\}$

and $\tilde{w} \propto \exp(imz)$

we finally obtain as solution for the perturbation δw:

$$\delta w \;\propto\; \exp\left\{-\frac{1}{2}\left[B(n_2-n_3)-\frac{n_1}{H_0}\right]\underline{z}\right\}\cdot\exp\{i(\underline{kx}+\underline{mz}+\sigma t)\} \qquad (34)$$

Free travelling wave in *x* and *z* with an amplitude changing exponentially with height!

Finished step 2 !!!

Step 3:

The remaining dependent variables are obtained from eqs. (25)-(29) by inserting

1.) $\hat{w}$ into $(29) \Rightarrow \hat{p}/\rho_0$

2.) $\hat{w}$ into $(26) \Rightarrow \hat{\Theta}$

3.) $\hat{p}/\rho_0$ into $(27) \Rightarrow \hat{u}$

4.) $\hat{p}/\rho_0$ into $(28) \Rightarrow \hat{v}$

5.) $\hat{u}$ and $\hat{w}$ into $(25) \Rightarrow \hat{\rho}/\rho_0$

Exact Solutions of the Linearized Equations

By setting the tracer parameters to 1 ($n_1 = n_2 = n_3 = n_4 = 1$) in the solution we have derived (i.e. in the dispersion relationship (33) and in the expression for δw (34)) we obtain directly the solution for the exact linearized equations:

From (33) with $B + g/c^2 = 1/H_0 \;\Rightarrow$

dispersion relationship for the exact linearized equations:

$$\boxed{m^2 = \frac{k^2(gB - \sigma^2)}{\sigma^2 - f^2} + \frac{\sigma^2}{c^2} - \frac{1}{4H_0^{\,2}}} \qquad (36)$$

From (34) =>

$$\boxed{\delta w \propto \exp\left\{\frac{z}{2H_0}\right\} \cdot \exp\left\{ i\,(kx + mz + \sigma t) \right\}} \qquad (36a)$$

Amplitude of exact solution grows exponentially with height.

Solutions of the dispersion relationship (36)

Re-arranging (36) gives a 4th order polynomial in σ:

$$\sigma^4 - \sigma^2\left[f^2 + c^2\left(k^2+m^2+\frac{1}{4H_0^2}\right)\right] + c^2\left[k^2 gB + f^2\left(m^2+\frac{1}{4H_0^2}\right)\right] = 0$$

4 solutions:

$$\sigma_g^2 = \frac{1}{2}\left[f^2+c^2\left(k^2+m^2+\frac{1}{4H_0^2}\right)\right]\left[1-\sqrt{1-\frac{4c^2\left[k^2gB+f^2\left(m^2+\frac{1}{4H_0^2}\right)\right]}{\left[f^2+c^2\left(k^2+m^2+\frac{1}{4H_0^2}\right)\right]^2}}\right] \qquad (38)$$

pair of inertial-gravity waves

$$\sigma_a^2 = \frac{1}{2}\left[f^2+c^2\left(k^2+m^2+\frac{1}{4H_0^2}\right)\right]\left[1+\sqrt{1-\frac{4c^2\left[k^2gB+f^2\left(m^2+\frac{1}{4H_0^2}\right)\right]}{\left[f^2+c^2\left(k^2+m^2+\frac{1}{4H_0^2}\right)\right]^2}}\right] \qquad (39)$$

pair of acoustic waves

Closer examination of the solutions (38) and (39)

1

By using the following inequalities, valid for typical values of the system parameters $f\,(\approx 10^{-4}\,s^{-1})$, $H_0\,(\approx 7km)$, g $(\approx 9.8\,m/s^2)$, $B\,(\approx 10^{-5}\,m^{-1})$ and $c\,(\approx 300m/s)$ in the atmosphere of the Earth, we can simplify expressions (38) and (39).

$$f^2 << gB << \frac{c^2}{H_0^2} \quad , \quad \frac{f^2}{gB} << \frac{gBH_0^2}{c^2} << 1 \qquad (40)$$

$$\sigma_{g,a}^2 = \frac{1}{2}\left[f^2 + c^2\left(k^2 + m^2 + \frac{1}{4H_0^2}\right)\right]\left[1 \mp \sqrt{1 - \underbrace{\frac{4c^2\left[k^2 gB + f^2\left(m^2 + \frac{1}{4H_0^2}\right)\right]}{\left[f^2 + c^2\left(k^2 + m^2 + \frac{1}{4H_0^2}\right)\right]^2}}_{\equiv X}}\,\right] \qquad (38)\ (39)$$

With (40) => $X << 1.$

Use Taylor expansion of $\sqrt{1-X}$ to first order in X around X= 0:

$$\sqrt{1-X} = 1 - \frac{X}{2} + O\left(X^2\right)$$

Closer examination of the solutions (38) and (39)

2

$$\sigma_{g,a}^{2} = \frac{1}{2}\left[f^{2}+c^{2}\left(k^{2}+m^{2}+\frac{1}{4H_0^{2}}\right)\right]\left[1\mp\sqrt{1-\underbrace{\frac{4c^{2}\left[k^{2}gB+f^{2}\left(m^{2}+\frac{1}{4H_0^{2}}\right)\right]}{\left[f^{2}+c^{2}\left(k^{2}+m^{2}+\frac{1}{4H_0^{2}}\right)\right]^{2}}}_{\equiv X}}\,\right] \qquad \begin{matrix}(38)\\(39)\end{matrix}$$

Replacing $\sqrt{1-X}$ by $1-\dfrac{X}{2}$ in (38) gives

$$\boxed{\sigma_g^{2}} \approx \frac{1}{2}\left[f^{2}+c^{2}\left(k^{2}+m^{2}+\frac{1}{4H_0^{2}}\right)\right]\frac{X}{2} \approx \boxed{\frac{k^{2}gB+f^{2}\left(m^{2}+\frac{1}{4H_0^{2}}\right)}{k^{2}+m^{2}+\frac{1}{4H_0^{2}}}} \qquad (38a)$$

Equation (39) simplifies to

$$\boxed{\sigma_a^{2} \approx c^{2}\left(k^{2}+m^{2}+\frac{1}{4H_0^{2}}\right)} \qquad (39a)$$

Closer examination of the simplified solution (38a) 1

$$\sigma_g^2 \approx \frac{g\underline{B}k^2 + \underline{f}^2\left(m^2 + \frac{1}{4H_0^2}\right)}{k^2 + m^2 + \frac{1}{4H_0^2}} \qquad (38a)$$

For $B = 0$ and $f = 0$ => $\sigma_g = 0$, i.e. in a system with zero static stability and no rotation these waves can't form!

=> Restoring forces (responsible for bringing the displaced air parcels back to the equilibrium location) for this wave type are the buoyancy force and the Coriolis force (inertial force). => These waves are called

inertial-buoyancy waves or, more commonly, inertial-gravity waves

Closer examination of the simplified solution (38a) 2

$$\sigma_g^2 \approx \frac{gBk^2 + f^2\left(m^2 + \frac{1}{4H_0^2}\right)}{k^2 + m^2 + \frac{1}{4H_0^2}} \qquad (38a)$$

Short wave limit:

For short waves in the horizontal (i.e. for large k) expression (38a) reduces to

$$\sigma_g^2 \approx \frac{gBk^2}{k^2 + m^2 + \frac{1}{4H_0^2}} \qquad (41)$$

No Coriolis parameter f in (41)!
These waves are too short to be (noticeably) modified by rotation,
i.e. pure (internal) gravity waves!
Restoring force is the buoyancy force.

These waves form only in stable stratification (for $B > 0$)!

For neutral stratification (B=0)
=> $\boxed{\sigma_g = 0}$, i.e. no waves!

For unstable stratification ($B < 0$)
=> $\boxed{\sigma_g}$ is imaginary,
no waves!

$$B \equiv \frac{\partial}{\partial z}(\ln \theta_0)$$

Closer examination of the simplified solution (38a) 3

$$\boxed{\sigma_g^2 \approx \frac{gB\,k^2}{k^2 + m^2 + \dfrac{1}{4H_0^2}}} \quad (41)$$

Very short wave limit:

$i.e.\ k$ such that $k^2 >> m^2 + 1/\left(4H_0^2\right)$

From (41) => $\boxed{\sigma_g = \pm\sqrt{gB}}$

$\sqrt{gB}$ is called buoyancy frequency or Brunt-Väisälä frequency (often denoted by N).

Buoyancy frequency is the upper limit to frequency of gravity waves!

Some properties of pure (internal) gravity waves 1

We neglect the term $1/\left(4H_0^2\right)$ in (41) for the following discussion.
This is equivalent to assuming that the basic state density does not change with z ($d\ln(\rho_0)/dz = 0$), i.e. the basic state is incompressible.

=> dispersion relationship of gravity waves in this type of fluid is

$$\sigma_g = \pm\frac{\sqrt{gB}\,k}{\sqrt{k^2+m^2}} \qquad (41a)$$

Slope of phase lines (= angle α to the local vertical $\vec{e}_z \equiv (0,1)$)

Wave vector $\vec{K} = (k,m)$ is perpendicular to phase lines, so $\vec{K}_\perp \equiv (-m,k)$ is parallel to the phase lines.

$$\Rightarrow \cos\alpha = \frac{\vec{K}_\perp \cdot \vec{e}_z}{|\vec{K}_\perp|} = \frac{k}{\sqrt{k^2+m^2}} = \frac{|\sigma_g|}{\sqrt{gB}}$$

$$\Rightarrow \begin{cases} \text{waves with } \sigma_g = \pm\sqrt{gB} \text{ have } \underline{\text{vertical}} \text{ phase lines} \\ \text{waves with small } \sigma_g \text{ have almost } \underline{\text{horizontal}} \text{ phase lines} \end{cases}$$

Some properties of pure (internal) gravity waves 2

Group velocity $\vec{c}_g$ is perpendicular to phase velocity $\vec{c}$!

$$\vec{c}_g \equiv \left(\frac{\partial \sigma}{\partial k}, \frac{\partial \sigma}{\partial m} \right) = \pm \sqrt{gB} \left(\frac{m^2}{\sqrt{(k^2+m^2)^3}}, \frac{-km}{\sqrt{(k^2+m^2)^3}} \right)$$

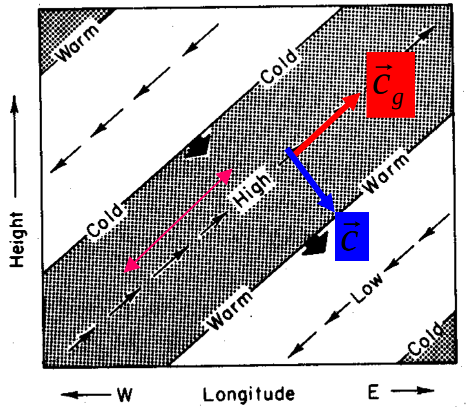
Since $\vec{c} \parallel \vec{K}$ and $\vec{K} \equiv (k,m) \perp \vec{c}_g$ => $\vec{c}_g \perp \vec{c}$.

Dispersive waves:

Horizontal and vertical phase speeds depend on the wave numbers.

Transversal waves: Particle path is parallel to the wave fronts.

From J.R.Holton:
An Introduction
to Dynamic
Meteorology



Idealized cross section for internal gravity wave showing phases of p, T & winds.

Example: Lee waves

Closer examination of the simplified solution (38a) 4

$$\sigma_g^2 \approx \frac{gBk^2 + f^2\left(m^2 + \frac{1}{4H_0^2}\right)}{k^2 + m^2 + \frac{1}{4H_0^2}} \qquad (38a)$$

Long inertial-gravity waves

These waves are influenced by the rotation of the earth.
Their frequency is given by (38a).

Long wave limit (k <<):

$$\sigma_g{}^2 \xrightarrow{k<<} f^2$$

Waves with $\sigma = \pm f$ are pure inertial waves.
(Not influenced by buoyancy force.)

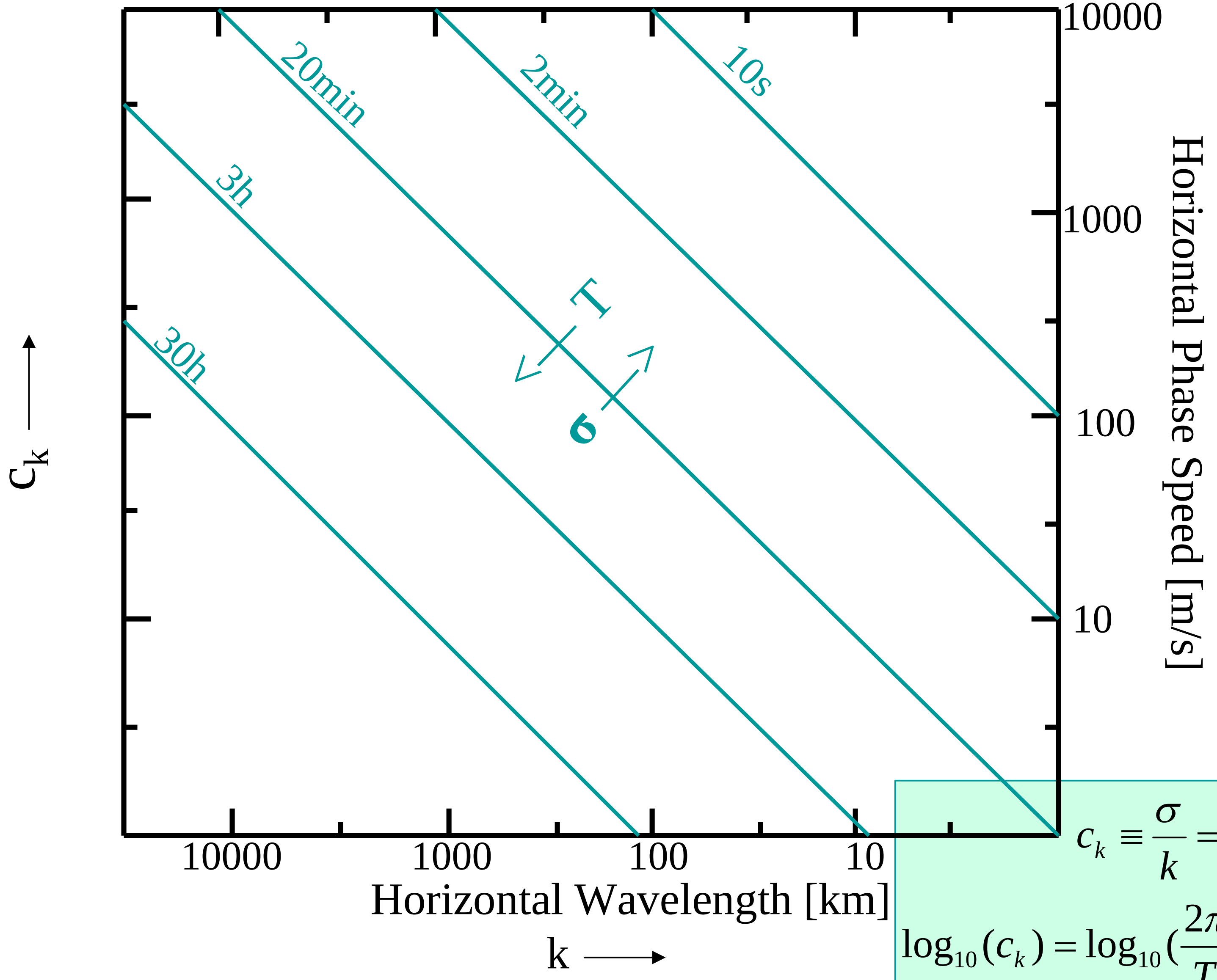
* Small frequency but large horizontal phase speeds!

$$|c_k| \equiv \frac{|\sigma|}{k} = \frac{f}{k} >> \quad (\text{for } k <<)$$

* Dispersive waves.

* Numerical nuisance because of their large phase speeds!

Dispersion Diagram $c_k(k)$

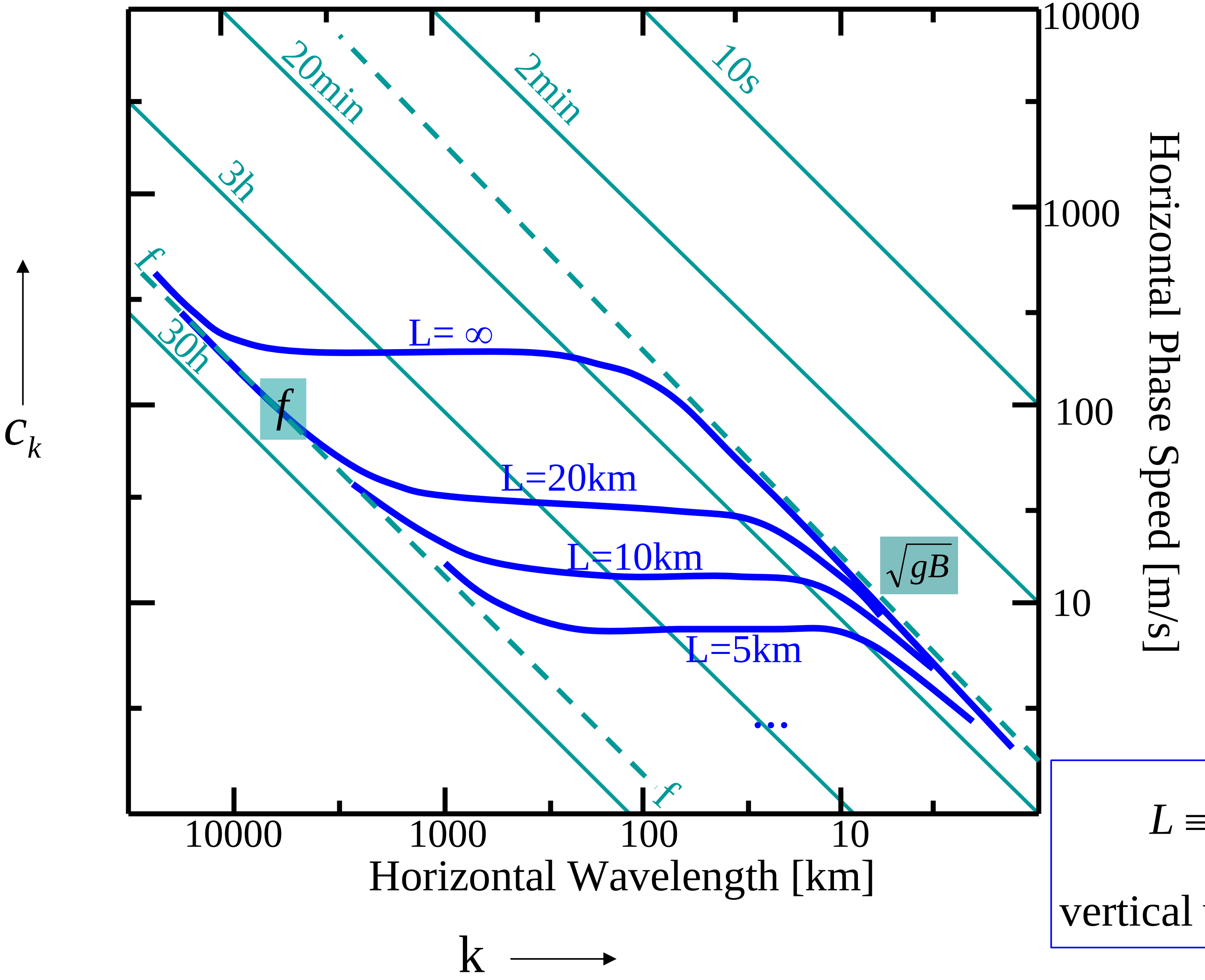


$$c_k \equiv \frac{\sigma}{k} = \frac{2\pi}{T}\cdot\frac{1}{k} \quad =>$$

$$\log_{10}(c_k) = \log_{10}\left(\frac{2\pi}{T}\right)\underline{-1}\cdot\log_{10}(k)$$

From J. S. A. Green: Dynamics lecture notes

Dispersion curves for inertial-gravity waves



$$L \equiv \frac{2\pi}{m}$$

vertical wavelength

Closer inspection of equation (39a): acoustic waves 1

$$\sigma_a^2 \approx c^2\left(k^2 + m^2 + \frac{1}{4H_0^2}\right) \quad (39a)$$

Here $c \equiv \sqrt{\gamma RT}$ is the adiabatic (Laplacian) speed of sound.

For very short waves in the horizontal (such that $k^2 >> m^2 + 1/\left(4H_0^2\right)$):

$\sigma_a \approx \pm ck$ => phase speed is the speed of sound!

These waves transmit pressure perturbations with the adiabatic speed of sound, i.e. type of waves with dispersion relationship (39a) are acoustic waves.

Very short acoustic waves

- * are non-dispersive, i.e. c_k is the same for all k.
- * have group velocity = phase velocity (in the horizontal)
- * are longitudinal waves (particle path is perpendicular to wave fronts)
- * have vertical phase lines (since $\cos\alpha = 1$), i.e. horizontal propagation.

$$\cos\alpha = \frac{k}{\sqrt{k^2 + m^2}} = \frac{ck}{|\sigma_a|} \quad \text{(neglected } \frac{1}{4H_0^2}\text{!)}$$

Closer inspection of equation (39a): acoustic waves 2

$$\sigma_a^2 \approx c^2\left(k^2 + m^2 + \frac{1}{4H_0^2}\right) \qquad (39a)$$

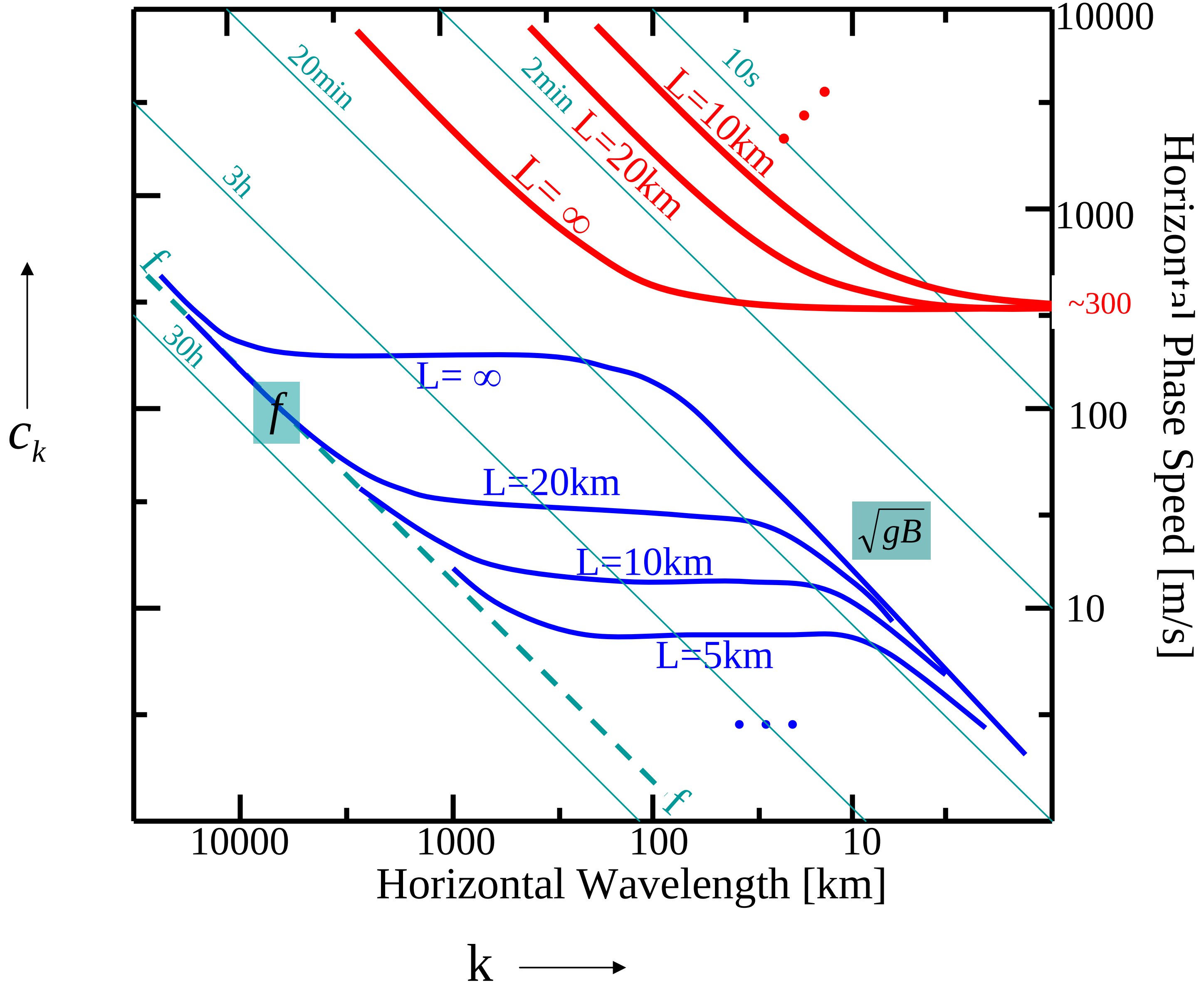
Long acoustic waves (long in the horizontal, i.e. very small k):

For $k \rightarrow 0$
$$\sigma_a^2 \approx c^2\left(m^2 + \frac{1}{4H_0^2}\right)$$

These long acoustic waves

- are dispersive
- have large horizontal phase speeds
- have almost horizontal phase lines ($\alpha \rightarrow 90^o$), i.e. almost vertical propagation

Dispersion curves of acoustic waves



Acoustic waves are a <u>numerical problem</u> because of their <u>high frequency</u> and <u>large phase speed</u>.

It would be good if we could filter them out of the system.

How?

By modifying the basic equation such that they don’t support this wave type.

Simplified solutions to the linearized equations: Filtering approximations

We will learn how to modify the linearized basic equations so that they don’t support acoustic waves and/or gravity waves anymore as solutions.
The physical principles behind these approximations can then be extended to achieve the same for the non-linear equations.

We will also investigate the impact the hydrostatic approximation to the pressure field has on the different wave types and determine conditions under which it is valid.

Now we will make use of the tracer parameters (n_1, n_2, n_3, n_4) we had introduced when we derived the solution of the linearized basic equations.

We will introduce approximations to the linearized basic equations (17)-(21) by setting individual tracers to 0 in the equations to eliminate the corresponding terms.
By setting these tracers to 0 also in the derived solutions we immediately obtain the solutions of the modified equations.

The elimination of acoustic waves (1)

Acoustic waves occur in any elastic medium.
Elastic compressibility is represented by $\frac{\partial \rho}{\partial t}$ in the continuity equation.

$$\frac{\partial \delta u}{\partial t} - f\delta \mathrm{v} + \frac{\partial}{\partial x}\left(\frac{\delta p}{\rho_0}\right) = 0 \quad (17)$$

$$\frac{\partial \delta \mathrm{v}}{\partial t} + f\delta u = 0 \quad (18)$$

$$n_4 \frac{\partial \delta w}{\partial t} + \frac{\partial}{\partial z}\left(\frac{\delta p}{\rho_0}\right) - n_3 B \frac{\delta p}{\rho_0} - g\delta\Theta = 0 \quad (19)$$

$$n_2 \boxed{\frac{\partial}{\partial t}\left(\frac{\delta \rho}{\rho_0}\right)} + \frac{\partial \delta u}{\partial x} + \frac{\partial \delta w}{\partial z} - n_1 \frac{\delta w}{H_0} = 0 \quad (20)$$

$$\frac{\partial \delta \Theta}{\partial t} + B\delta w = 0 \quad (21)$$

This term can be removed from (20) by setting $n_2 = 0$.

Setting $n_2 = 0$ in (33) immediately gives the dispersion expression for the modified set of equations.

But we have to be careful!!!

The elimination of acoustic waves (2)

In eq. (32) n_2 and n_3 occur in the combination $(n_2 - n_3)$ which vanishes in the exact equation (i.e. when $n_1 = n_2 = n_3 = n_4 = 1$).

$$\sigma\left\{\frac{d^2}{dz^2} + \left[B\boxed{(n_2 - n_3)} - \frac{n_1}{H_0}\right]\frac{d}{dz} + (gB - n_4\sigma^2)\left(\frac{k^2}{\sigma^2 - f^2} - \frac{n_2}{c^2}\right) - Bn_3\left(Bn_2 - \frac{n_1}{H_0}\right)\right\}\hat{w}(z) = 0 \qquad (32)$$

=> A spurious term will arise in (32) if we set n_2 to zero but not n_3 or vice-versa!

=> **We have to set always $n_2 = n_3$!**

Anelastic approximation is $n_2=0$ & $n_3=0$!

$$\frac{\partial \delta u}{\partial t} - f\delta \mathrm{v} + \frac{\partial}{\partial x}\left(\frac{\delta p}{\rho_0}\right) = 0 \qquad (17)$$

$$\frac{\partial \delta \mathrm{v}}{\partial t} + f\delta u = 0 \qquad (18)$$

$$n_4\frac{\partial \delta w}{\partial t} + \frac{\partial}{\partial z}\left(\frac{\delta p}{\rho_0}\right) - n_3 B\boxed{\frac{\delta p}{\rho_0}} - g\delta\Theta = 0 \qquad (19)$$

$$\boxed{n_2\frac{\partial}{\partial t}\left(\frac{\delta p}{\rho_0}\right)} + \frac{\partial \delta u}{\partial x} + \frac{\partial \delta w}{\partial z} - n_1\frac{\delta w}{H_0} = 0 \qquad (20)$$

$$\frac{\partial \delta\Theta}{\partial t} + B\delta w = 0 \qquad (21)$$

The elimination of acoustic waves (3)

Setting $n_2 = n_3 = 0$ and $n_1 = 1 = n_4$ in (33)

$$m^2 = \left(gB - \enclose{circle}{n_4}\sigma^2\right)\left(\frac{k^2}{\sigma^2 - f^2} - \frac{\enclose{circle}{n_2}}{c^2}\right) - B\enclose{circle}{n_3}\left(B\enclose{circle}{n_2} - \frac{\enclose{circle}{n_1}}{H_0}\right) - \frac{1}{4}\left[B\left(\enclose{circle}{n_2} - \enclose{circle}{n_3}\right) - \frac{\enclose{circle}{n_1}}{H_0}\right]^2 \qquad (33)$$

=> $$m^2 = (gB - \sigma^2)\frac{k^2}{\sigma^2 - f^2} - \frac{1}{4H_0^2} \qquad (46)$$

This dispersion relation has only 2 roots in σ not 4 as (33) => only one wave type left!

(46) <=> $$\sigma^2 = \frac{gBk^2 + f^2\left(m^2 + \frac{1}{4H_0^2}\right)}{k^2 + m^2 + \frac{1}{4H_0^2}} \qquad (47)$$

This is identical to the dispersion relation for inertial-gravity waves (38a)!

No acoustic waves!

Consequently:

Acoustic waves have been eliminated!

Inertial-gravity waves are not distorted by the anelastic approximation.

The elimination of acoustic waves (4)

Under what conditions is it OK to make the anelastic approximation?

Compare (46) with the exact dispersion relation (36):

(46) $$m^2 = \left(gB - \sigma^2\right)\left(\frac{k^2}{\sigma^2 - f^2}\right) - \frac{1}{4H_0^2}$$

$$m^2 = \frac{k^2(gB - \sigma^2)}{\sigma^2 - f^2} + \frac{\sigma^2}{c^2} - \frac{1}{4H_0{}^2}$$ (36)

When can we neglect $\frac{\sigma^2}{c^2}$ in (36)?

Re-arrange (36): $$k^2 + m^2 + \frac{1}{4H_0^2} + \frac{k^2\left(f^2 - gB\right)}{\sigma^2 - f^2} - \frac{\sigma^2}{c^2} = 0$$

$\frac{\sigma^2}{c^2}$ can be neglected in (36) if $$\sigma^2 << \underbrace{c^2\left(k^2 + m^2 + \frac{1}{4H_0^2}\right)}_{= \sigma_a^2 \text{ (39a)}}$$

The elimination of acoustic waves (5)

=> It is OK to make the anelastic approximation if the frequencies of the remaining waves are much smaller than the acoustic frequency.

This condition is satisfied for inertial-gravity waves!

=> Acoustic filtered equations can be used with confidence for a detailed study of inertial-gravity waves in the atmosphere (e.g. for modelling of mountain gravity waves).

The hydrostatic approximation

1

Questions we are going to address:

How does the hydrostatic approximation to the pressure field affect the inertial-gravity waves and the acoustic waves?

When is it alright to make this approximation?

Hydrostatic approximation = neglect of the vertical acceleration Dw/Dt in vertical momentum equation (3).

$$\cancel{\frac{Dw}{Dt}} + g = -\frac{1}{\rho}\frac{\partial p}{\partial z} \qquad (3)$$

In the linearized momentum eq. (19) the vertical acceleration is represented by $\partial \delta w / \partial t$ (since we assumed the basic state to be at rest). => set $n_4=0$!

$$n_4 \boxed{\frac{\partial \delta w}{\partial t}} + \frac{\partial}{\partial z}\left(\frac{\delta p}{\rho_0}\right) - n_3 B \frac{\delta p}{\rho_0} - g\delta\Theta = 0 \qquad (19)$$

The hydrostatic approximation 2

Validity criterion:

From the dispersion relation (33)

$$m^2 = \left(gB - \underline{n_4}\sigma^2\right)\left(\frac{k^2}{\sigma^2 - f^2} - \frac{n_2}{c^2}\right) - Bn_3\left(Bn_2 - \frac{n_1}{H_0}\right) - \frac{1}{4}\left[B\left(n_2 - n_3\right) - \frac{n_1}{H_0}\right]^2 \qquad (33)$$

we see that the term containing n_4 can be neglected if $\boxed{\sigma^2 << gB}$

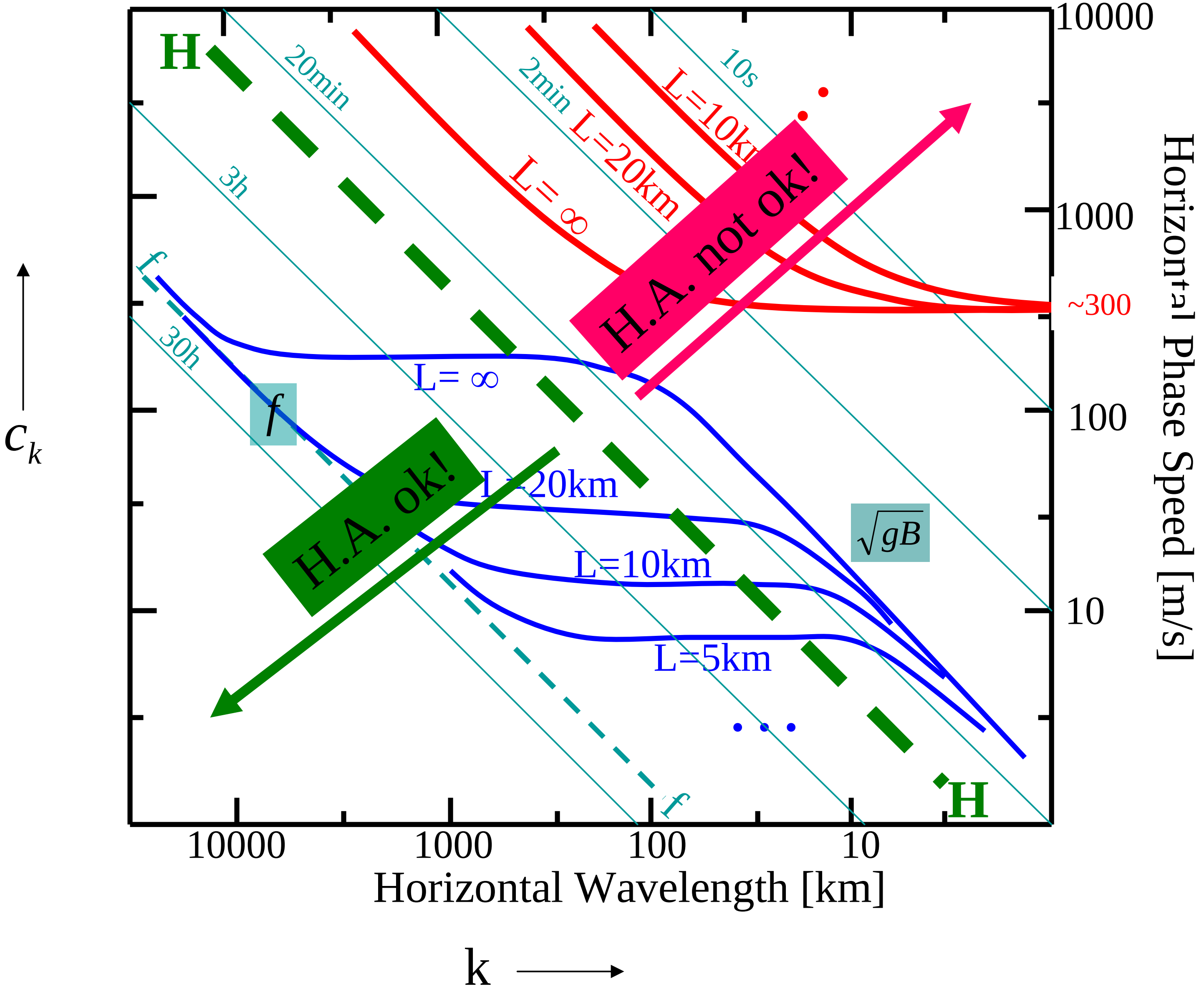
=> Hydrostatic approximation is OK for waves with frequencies much smaller the the buoyancy frequency!

This condition is

- * satisfied for inertial waves ($f^2 << gB$)
- * not satisfied for very short gravity waves $\left(\sigma_g^2 \xrightarrow{k \gg} gB \right)$
- * not satisfied for acoustic waves $\left(\sigma_a^2 > gB \quad \forall k, m! \right)$

=> Hydrostatic approximation affects acoustic waves and very short gravity waves.
Inertial waves and long gravity waves are unaffected.

Validity domain of the hydrostatic approximation (H.A.)



The hydrostatic approximation 3

Dispersion relationship in hydrostatic system:

Setting $n_4=0$ and $n_1=n_2=n_3=1$ in (33) and using $B+g/c^2=1/H_0$ gives:

$$\sigma^2=\frac{gBk^2+f^2\left(m^2+\frac{1}{4H_0^2}\right)}{m^2+\frac{1}{4H_0^2}} \qquad (49)$$

$$\sigma_g^2\approx\frac{gBk^2+f^2\left(m^2+\frac{1}{4H_0^2}\right)}{\underline{k^2}+m^2+\frac{1}{4H_0^2}} \qquad (38a)$$

(49) looks like the dispersion relation for inertial-gravity waves (38a) but k^2 is missing in the denominator!

Consequently, inertial-gravity waves will be distorted in the hydrostatic pressure field unless $k^2 \ll m^2+1/(4H_0^2)$!

=> Hydrostatic approximation <u>should not be used</u> if horizontal and vertical length scales in the system are of comparable magnitude ($L_X \sim L_Z$).
(For example in convective scale models.)
It is <u>OK</u> for $L_X > L_Z$. (For example if $L_X \geq 100$km and $L_Z \sim 10$km.)

The hydrostatic approximation

4

(49) represents only <u>one pair of waves</u>
and these are (distorted) inertial-gravity waves.

$$\sigma^2 = \frac{gBk^2 + f^2\left(m^2 + \frac{1}{4H_0^2}\right)}{m^2 + \frac{1}{4H_0^2}} \qquad (49)$$

Acoustic waves seem to have been filtered out by making the hydrostatic approximation.

In fact, only acoustic waves for which $\delta w \not\equiv 0$ have been filtered out! These are all vertically propagating sound waves (i.e. with $m > 0$).

Waves with $\delta w \equiv 0$ are not represented by (49), because we had assumed $\hat{w} \not\equiv 0$ when we derived the dispersion relationship (33) (on which (49) is based) from the differential equation (32)!

Are there any (acoustic) waves with $\delta w \equiv 0$?

Yes!

Slide 24

Solutions of equation (32) 1

$$\sigma\left\{\frac{d^2}{dz^2}+\left[B(n_2-n_3)-\frac{n_1}{H_0}\right]\frac{d}{dz}+(gB-n_4\sigma^2)\left(\frac{k^2}{\sigma^2-f^2}-\frac{n_2}{c^2}\right)-Bn_3\left(Bn_2-\frac{n_1}{H_0}\right)\right\}\hat{w}(z)=0 \quad (32)$$

Solution 1: $\boxed{\sigma=0}$ Not a wave!

Inserting $\sigma = 0$ into (22)-(26) gives for the winds:

$$\hat{u}=\hat{w}=0 \text{ and } \hat{v}=\frac{ik}{f}\frac{\hat{p}}{\rho_0} \Rightarrow \delta v=\frac{1}{f\rho_0}\frac{\partial}{\partial x}\delta p$$, i.e. geostrophic motion.

Solution 2: $\boxed{\hat{w}\equiv 0 \ \forall z}$ Lamb wave

This solution will be discussed later in detail.

Further solutions: For $\sigma\neq 0$ and $\hat{w}\neq 0\ \forall z$ we have to solve

$$\frac{d^2\hat{w}}{dz^2}+\left[B(n_2-n_3)-\frac{n_1}{H_0}\right]\frac{d\hat{w}}{dz}+\left[(gB-n_4\sigma^2)\left(\frac{k^2}{\sigma^2-f^2}-\frac{n_2}{c^2}\right)-Bn_3\left(Bn_2-\frac{n_1}{H_0}\right)\right]\hat{w}=0 \quad (32a)$$

The Lamb wave

1

We examine now the case $\hat{w} \equiv 0, \ \forall z.$

In this case (32) is redundant. To obtain the frequencies of possible waves for this case we have to go back to eqs (29) and (31) and insert $\hat{w} \equiv 0.$

$$\frac{d}{dz}\hat{w} + \left(Bn_2 - \frac{n_1}{H_0} \right)\hat{w} + i\sigma\left(\frac{n_2}{c^2} - \frac{k^2}{\sigma^2 - f^2} \right)\frac{\hat{p}}{\rho_0} = 0 \quad (29)$$

$$i\sigma\frac{d}{dz}\left(\frac{\hat{p}}{\rho_0} \right) - i\sigma Bn_3\frac{\hat{p}}{\rho_0} + (gB - n_4\sigma^2)\hat{w} = 0 \quad (31)$$

$$\Rightarrow \begin{cases} \sigma\left(\dfrac{n_2}{c^2} - \dfrac{k^2}{\sigma^2 - f^2} \right)\dfrac{\hat{p}}{\rho_0} = 0 & (50) \\[2ex] \sigma\left(\dfrac{\partial}{\partial z} - Bn_3 \right)\dfrac{\hat{p}}{\rho_0} = 0 & (51) \end{cases}$$

Solution 1: $\dfrac{\hat{p}}{\rho_0} \equiv 0$ is a <u>trivial solution</u> because with $\hat{w} \equiv 0$ =>

$\hat{u} \equiv \hat{v} \equiv \hat{\Theta} \equiv \hat{\rho} \equiv 0$, i.e. perturbations of all variables vanish.

Solution 2: $\sigma = 0$, which is the geostrophic mode mentioned previously.

Wave solutions we obtain from

$$\begin{cases} \dfrac{n_2}{c^2} - \dfrac{k^2}{\sigma^2 - f^2} = 0 & (52a) \\[2ex] \left(\dfrac{\partial}{\partial z} - Bn_3 \right)\dfrac{\hat{p}}{\rho_0} = 0 & (52b) \end{cases}$$

The Lamb wave 2

$$\frac{n_2}{c^2} - \frac{k^2}{\sigma^2 - f^2} = 0 \quad \text{(52a)} \qquad \Leftrightarrow \qquad n_2\sigma = n_2 f^2 + c^2k^2 \quad \text{(52c)}$$

$$\left(\frac{\partial}{\partial z} - Bn_3\right)\frac{\hat{p}}{\rho_0} = 0 \quad \text{(52b)}$$

In an anelastic system ($n_2=n_3=0$) these waves can’t form.
They are a form of acoustic waves.

Dispersion relationship: (from (52c) by setting $n_2=n_3=1$)

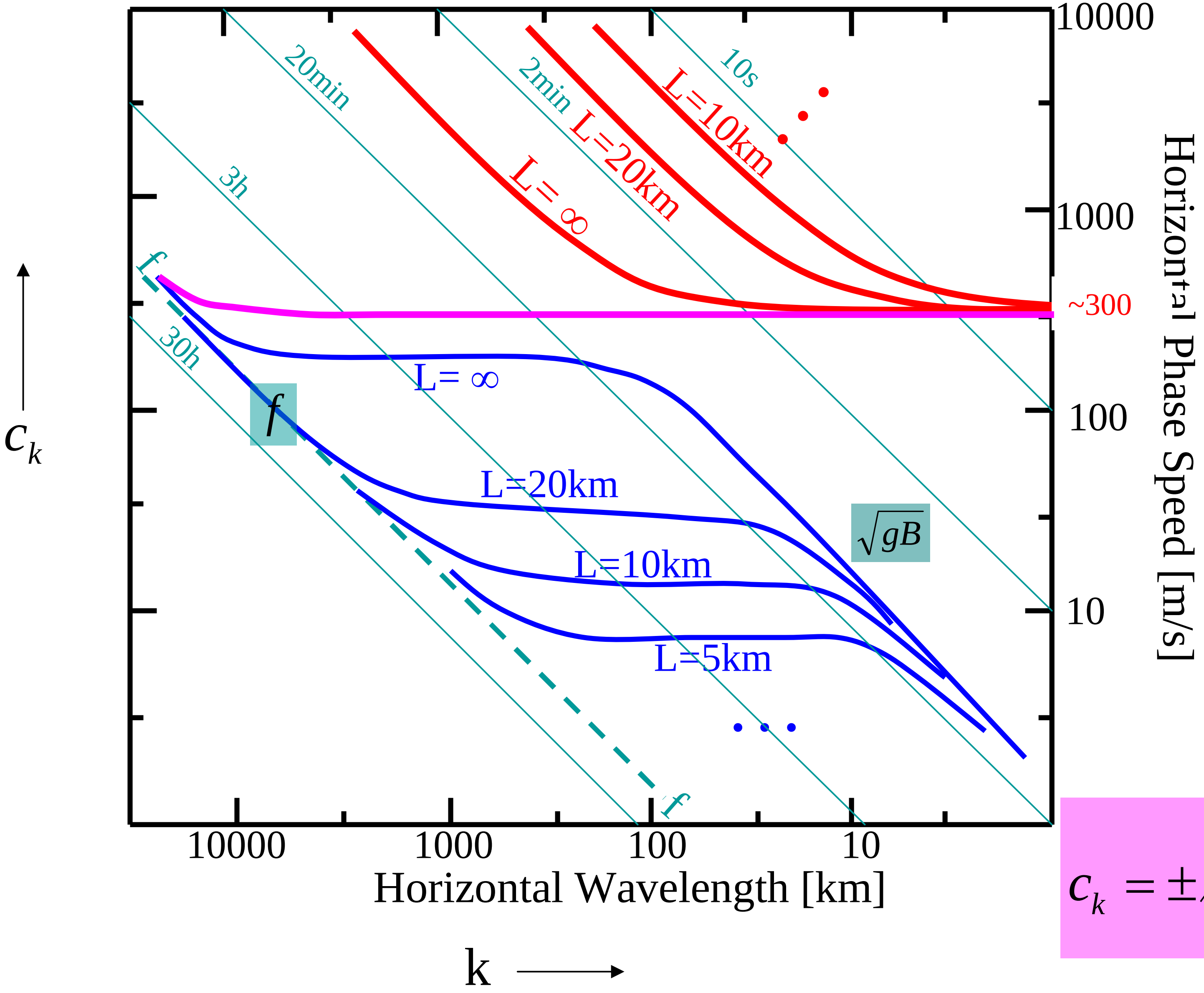
$$\sigma^2 = f^2 + c^2k^2 \quad \text{(53)}$$

Phase speed:

* for short waves: $c_k \approx \pm c$ (like for very short acoustic waves)

* for very long waves: $c_k \approx \pm \frac{f}{k}$ (like for inertial waves)

Dispersion curve of the Lamb wave



$$c_k = \pm\sqrt{\frac{f^2}{k^2} + c^2}$$

The Lamb wave 3

$$\left(\frac{\partial}{\partial z} - B\enclose{circle}{n_3}\right)\frac{\hat{p}}{\rho_0} = 0 \qquad \text{(52b)}$$

Structure of the Lamb wave:

From (52b) with $n_3=1$ => $\frac{\hat{p}}{\rho_0} = e^{Bz}$ => $\frac{\delta p}{\rho_0} \propto e^{Bz} \cdot e^{i(kx+\sigma t)}$

Wave only in x!

And from equations (22)-(26) with $\hat{w} \equiv 0$ =>

$$\delta u = -\frac{\sigma}{kc^2}\left(\frac{\delta p}{\rho_0}\right), \quad \frac{\delta\rho}{\rho_0} = \frac{1}{c^2}\left(\frac{\delta p}{\rho_0}\right), \quad \delta v \equiv \delta w \equiv \delta\Theta \equiv 0$$

This wave is a pressure perturbation propagating only horizontally (m=0).

* Lamb waves have been observed in the atmosphere after violent explosions like volcanic eruptions and atmospheric nuclear tests.

* They are of negligible physical significance.

* Can be suppressed by anelastic approximation. However, they are not more of a numerical problem than long inertial-gravity waves since largest phase speeds are comparable to the phase speeds of inertial waves.

Filtering of gravity waves 1

Gravity waves were filtered out in older models of the large-scale dynamics of the atmosphere because they can cause numerical problems (long waves have large phase speeds!). They are not filtered out in modern models anymore.

Filter: Set local rate of change of divergence to zero => no gravity waves!

$$\frac{\partial \delta u}{\partial t} - f\delta v + \frac{\partial}{\partial x}\left(\frac{\delta p}{\rho_0}\right) = 0 \quad (17)$$

$$\frac{\partial \delta v}{\partial t} + f\delta u = 0 \quad (18)$$

$$n_4 \frac{\partial \delta w}{\partial t} + \frac{\partial}{\partial z}\left(\frac{\delta p}{\rho_0}\right) - n_3 B \frac{\delta p}{\rho_0} - g\delta\Theta = 0 \quad (19)$$

$$n_2 \frac{\partial}{\partial t}\left(\frac{\delta \rho}{\rho_0}\right) + \frac{\partial \delta u}{\partial x} + \frac{\partial \delta w}{\partial z} - n_1 \frac{\delta w}{H_0} = 0 \quad (20)$$

$$\frac{\partial \delta\Theta}{\partial t} + B\delta w = 0 \quad (21)$$

We demonstrate this on a simplified system:

1.) Start from linearized basic equations (17)-(21).

2.) Make hydrostatic approximation (n_4=0!)

3.) Filter out all acoustic waves by making the anelastic approximation (n_2=n_3=0!)

4.) Assume the atmosphere to be incompressible, i.e. $\frac{\partial \rho_0}{\partial z} = 0 \Leftrightarrow \frac{1}{H_0} \equiv -\frac{1}{\rho_0}\frac{\partial \rho_0}{\partial z} = 0 \Leftrightarrow$ n_1=0!

5.) Eliminate δΘ and δw by simple algebraic manipulations of (19)-(21).

=> system of 3 equations in the unknowns δu, δv, $\delta p/\rho_0$.

Filtering of gravity waves 2

=>

$$
\begin{aligned}
&\frac{\partial \delta u}{\partial t} - f\delta \mathrm{v} + \frac{\partial}{\partial x}\left(\frac{\delta p}{\rho_0}\right) = 0 \\
&\frac{\partial \delta \mathrm{v}}{\partial t} + f\delta u = 0 \\
&\frac{\partial}{\partial t}\frac{\partial^2}{\partial z^2}\left(\frac{\delta p}{\rho_0}\right) - gB\frac{\partial \delta u}{\partial x} = 0
\end{aligned}
$$

We need to introduce the divergence $D \equiv \frac{\partial u}{\partial x} + \frac{\partial \mathrm{v}}{\partial y}$ into these equations.

$D = \frac{\partial \delta u}{\partial x}$, since $\frac{\partial \delta \mathrm{v}}{\partial y} \equiv 0$ because of $\frac{\partial}{\partial y} \equiv 0$!

Therefore, apply $\frac{\partial}{\partial x}$ to momentum equations (first two equations).

=>

$$
\begin{aligned}
&\enclose{circle}{n_5}\,\frac{\partial \underline{D}}{\partial t} - f\frac{\partial \delta \mathrm{v}}{\partial x} + \frac{\partial^2}{\partial x^2}\left(\frac{\delta p}{\rho_0}\right) = 0 \\
&\frac{\partial}{\partial t}\frac{\partial \delta \mathrm{v}}{\partial x} + f\underline{D} = 0 \\
&\frac{\partial}{\partial t}\frac{\partial^2}{\partial z^2}\left(\frac{\delta p}{\rho_0}\right) - gB\underline{D} = 0
\end{aligned}
$$

New tracer n_5 for local rate of change of divergence.

Filtering of gravity waves 3

These equations have the dispersion relationship

$$\sigma\left[f^2 - \textcircled{n_5}\sigma^2 + gB\frac{k^2}{m^2}\right] = 0 \qquad (54)$$

For $n_5=1$ => $$\sigma^2 = f^2 + gB\frac{k^2}{m^2}$$

Compare to dispersion relation in the hydrostatic system (49):

$$\sigma^2 = \frac{gBk^2}{m^2 + \frac{1}{4H_0^2}} + f^2 \qquad (49)$$

These waves are (distorted) inertial-gravity waves!

(Distorted because of hydrostatic approximation and incompressibility approx. $n_1=0$)

Setting $n_5= 0$ eliminates this inertial-gravity wave solution!

=>

Suppressing local rate of change of divergence “kills” the inertial-gravity waves!

Filtered Rossby Waves (Planetary Waves)

1

This was previously the $\boldsymbol{\sigma} \equiv 0$ solution.

Now we let the field vary in meridional direction too, i.e. $\dfrac{\partial}{\partial y} \not\equiv 0!$

=> back to 3-dimensional system (x,y,z)
Coriolis parameter f varies now with latitude.

To simplify the problem:

- * make anelastic approximation
- * make hydrostatic approximation
- * set local rate of change of divergence = 0

=> no acoustic and no gravity waves!

- * set n_1=0 (i.e. incompressible atmosphere)
- * make β-plane approximation to f:

$$f(y) \approx f(y_0) + \frac{\partial f}{\partial y}(y_0)\cdot(y-y_0) \equiv f_0 + \beta(y-y_0)$$

Filtered Rossby Waves 2

With these approximations one obtains from the linearized 3d basic equations the following equation for the pressure perturbation $\frac{\delta p}{\rho_0} \equiv \tilde{p}$

$$\frac{\partial}{\partial t}\left(\frac{\partial^2 \tilde{p}}{\partial x^2} + \underline{\frac{\partial^2 \tilde{p}}{\partial y^2}} + \frac{f_0^2}{gB}\frac{\partial^2 \tilde{p}}{\partial z^2}\right) + \beta\frac{\partial \tilde{p}}{\partial x} = 0 \qquad (55)$$

Try waves as solutions: $\tilde{p} \propto \exp\left[i\left(kx + \underline{ly} + mz \enclose{circle}{-} \sigma t\right)\right]$

Inserting into (55) gives dispersion relationship:

$$\sigma = \enclose{circle}{-}\frac{\beta k}{k^2 + \underline{l^2} + m^2 \frac{f_0^2}{gB}}$$

For $\beta = 0$ => $\sigma = 0$!
So this wave type can only exist when the Coriolis parameter varies with latitude!

Filtered Rossby Waves 3

$$\sigma = -\frac{\beta k}{k^2 + l^2 + m^2 \frac{f_0^2}{gB}}$$

Rossby waves don’t occur in pairs of eastward and westward propagating waves, as do acoustic waves and inertial-gravity waves.
There are only westward propagating Rossby waves!
(westward relative to the mean zonal flow)

$$c_k = +\frac{\sigma}{k} = -\frac{\beta}{k^2 + l^2 + m^2 \frac{f_0^2}{gB}}$$

So far we have always assumed the mean flow to be zero. For a constant basic zonal flow u_0 the frequency observed at the ground is the Doppler-shifted frequency $\sigma' = \sigma + u_0 k$ =>

Frequency observed at the ground:

$$\sigma' = u_0 k - \frac{\beta k}{k^2 + l^2 + m^2 \frac{f_0^2}{gB}}$$

=> zonal phase speed observed at the ground is:

$$c_x = u_0 - \frac{\beta}{k^2 + l^2 + m^2 \frac{f_0^2}{gB}}$$

=>

Rossby waves propagate westward relative to the mean flow. Relative to the ground they usually move eastward (when $u_0>0$ and $u_0>|c_k|$). They become stationary relative to the ground (i.e. $c_x = 0$) if k, l and m fulfill the condition

$$k^2 + l^2 + m^2 f_0^2/(gB) = \beta / u_0$$

Filtered Rossby Waves

4

$$\sigma' = u_0 k - \frac{\beta k}{k^2 + l^2 + m^2 \frac{f_0^2}{gB}}$$

Rossby waves are <u>dispersive</u>:

- <u>long</u> zonal and meridional waves are fastest!
- for short zonal Rossby waves such that $k^2 >> l^2 + m^2 f_0^2 / (gB)$

phase velocity:

$$c_x = u_0 - \frac{\beta}{k^2}$$

group velocity:

$$c_g = u_0 + \frac{\beta}{k^2}$$

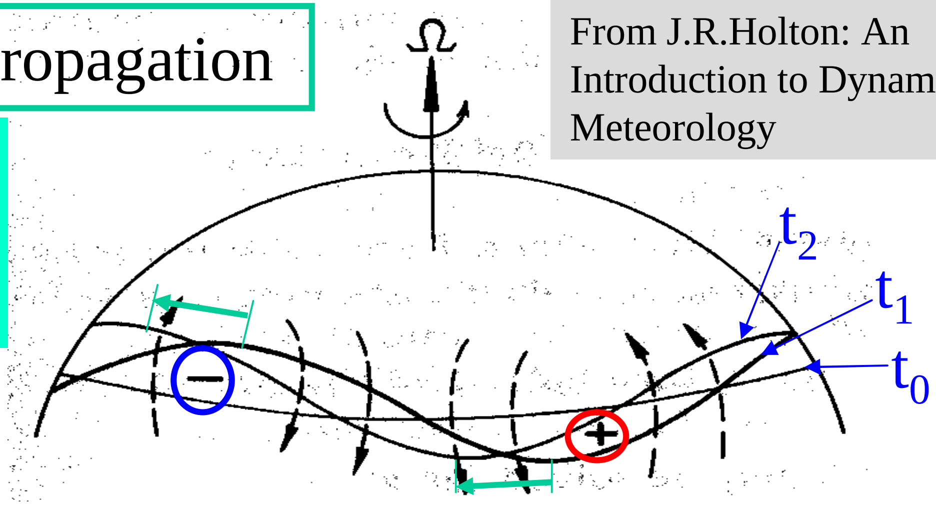
=> Group velocity is opposite in direction to phase velocity!

(with respect to the mean flow u_0)

Rossby waves pose <u>no</u> numerical problem because they have quite large periods (of the order of days) and don't move very fast (typically with ~10m/s).

Rossby Wave Propagation

A Rossby wave is a periodic vorticity field which propagates westward and conserves absolute vorticity.



From J.R.Holton: An Introduction to Dynamic Meteorology

Perturbation vorticity field and induced velocity field (dashed arrows) for a meridionally displaced chain of fluid parcels. Heavy wavy line shows original perturbation position, light line westward displacement of the pattern due to advection by the induced velocity.

Absolute vorticity η is given by $\eta = \zeta + f$,where ζ is the relative vorticity and f the Coriolis parameter.

Assume that at time t_0:

$\zeta=0 \Rightarrow \eta_0 = f_0$.

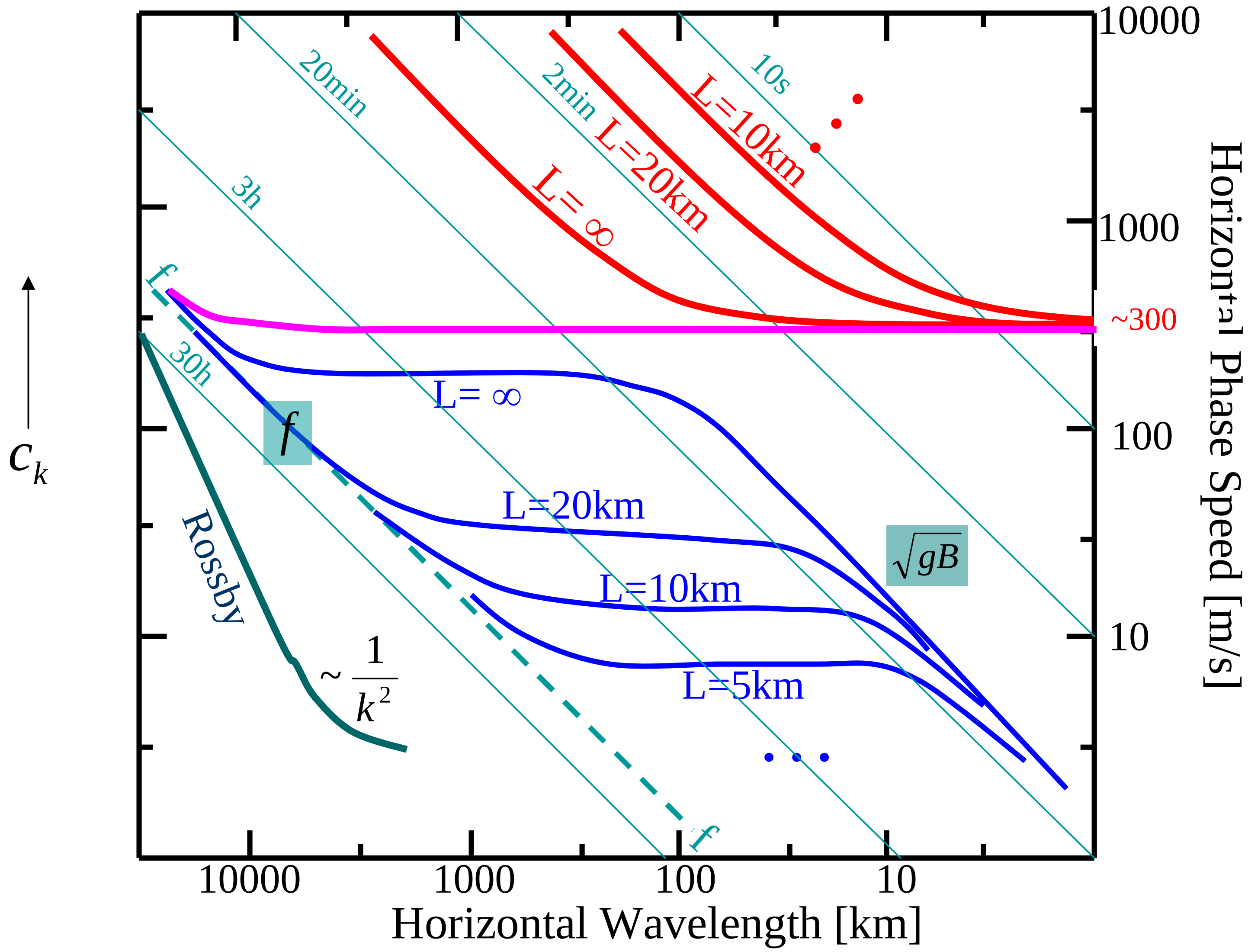
At t_1 we have a meridional displacement δy of a fluid parcel:

$\eta_1 = \zeta_1 + f_1 = f_0$ (because of conservation of absolute vorticity!)

$$\Rightarrow \ \zeta_1 = f_0 - f_1 = -\frac{\partial f}{\partial y}\delta y = -\beta\delta y \ \Rightarrow \begin{cases} \zeta_1>0 \text{ for } \delta y<0\text{, i.e. cyclonic for southward displacement} \\ \zeta_1<0 \text{ for } \delta y>0\text{, i.e. anticyclonic for northward displacement} \end{cases}$$

Meridional gradient of f resists meridional displacement and provides the restoring mechanism for Rossby waves.

Dispersion curve of Rossby wave

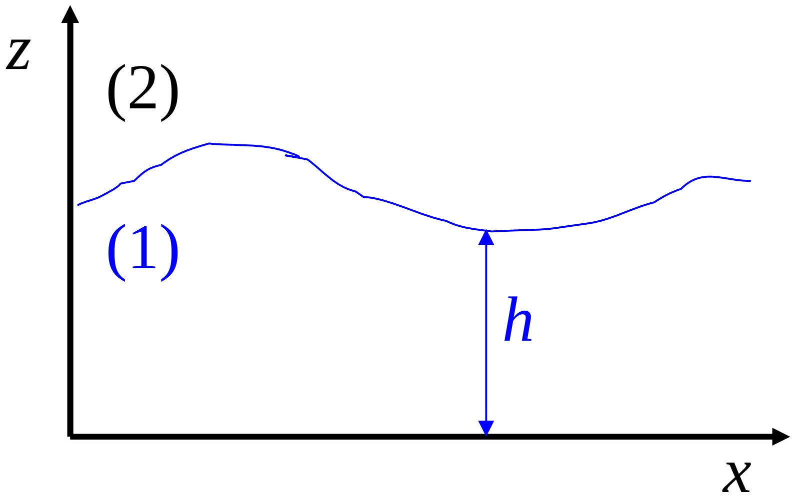


Rossby waves are not distorted by the hydrostatic approximation.

Surface Gravity Waves 1

Surface waves are waves on a boundary between two media.

Examples: - waves on air-water interface
- waves on an inversion



free surface $h(x,t)$

rigid boundary at $z = 0$

We work again only in 2d: (x,z) , i.e. neglect again variations in y $\left(\frac{\partial}{\partial y} \equiv 0!\right)$

Boundaries constrain our system now:

- rigid horizontal boundary at the bottom ($z=0$)
- free surface above (at $z=h(x,t)$)

“<u>Free surface</u>”: surface shape is free to respond to the motion within the fluid and is not known ‘a priori’.

The <u>fluid motion</u> and the <u>boundary shape</u> must be determined <u>simultaneously</u>.

Surface Gravity Waves 2

Equations governing the motion inside the fluid layer:

We study only waves with small amplitude (small perturbations), i.e. linearized equations can be used. We use equation (32a):

$$\frac{d^2\hat{w}}{dz^2}+\left[B(n_2-n_3)-\frac{n_1}{H_0}\right]\frac{d\hat{w}}{dz}+\left[(gB-n_4\sigma^2)\left(\frac{k^2}{\sigma^2-f^2}-\frac{n_2}{c^2}\right)-Bn_3\left(Bn_2-\frac{n_1}{H_0}\right)\right]\hat{w}=0 \tag{32a}$$

Simplifications / approximations:

1.) Assume unstratified fluid, i.e. static stability $B = 0$ => (32a) reduces to

$$\frac{d^2\hat{w}}{dz^2}-\frac{\enclose{circle}{n_1}}{H_0}\frac{d\hat{w}}{dz}-\enclose{circle}{n_4}\sigma\left(\frac{k^2}{\sigma^2-f^2}-\frac{\enclose{circle}{n_2}}{c^2}\right)\hat{w}=0 \tag{57}$$

2.) Filter out acoustic waves, i.e. set $n_2 = 0$ in (57), but only after we have seen how this affects the equations of the boundary conditions!!!

3.) Assume the pressure to be constant above the free surface.

No hydrostatic approximation at this stage! Carry n_4 along for future use.
No incompressibility approximation at this stage either! Carry n_1 along as well.

Surface Gravity Waves 3

Boundary conditions:

a.) Particles cannot cross the boundary between the two fluids!

b.) The two fluids must stay together and not separate at their common boundary. This is ensured by imposing continuity of the velocity component perpendicular to the boundary across the boundary.

Conditions a.) and b.) state that particles adjoining the surface follow the surface contour, i.e. the surface is a material boundary.

These are <u>kinematic conditions</u>. We need a <u>dynamic condition</u> too.

c.) The pressure in the two fluids must be equal at the common boundary (continuity of pressure).

Expressed in mathematical terms:

a.) & b.) $\frac{D}{Dt}(z - h(x,t)) = 0$ at boundary $z = h(x,t)$ => $w(h) = \frac{Dh}{Dt}$

At a <u>flat and rigid boundary</u> (*h*=constant in space and time) $\underline{w \equiv 0}$.

Surface Gravity Waves

4

Expressed in mathematical terms (continuation):

c.) Continuity of pressure: $p_1 = p_2$ at $z = h(x,t)$.

$$\frac{Dp_1}{Dt} = \frac{Dp_2}{Dt} \text{ at } z = h(x,t)$$

p_2 we assume to be constant in space and time.

$$\Rightarrow \boxed{\frac{Dp_1}{Dt} = 0 \quad \text{at} \quad z = h(x,t).}$$

We will call p_1 simply p from now on.

=> The following set of <u>non-linear</u> boundary conditions has to be fulfilled:

(i) $\boxed{w = 0 \quad \text{at} \quad z = 0}$

(ii) $\boxed{w = \frac{Dh}{Dt} \quad \text{at} \quad z = h(x,t)}$

(iii) $\boxed{\frac{Dp}{Dt} = 0 \quad \text{at} \quad z = h(x,t)}$

Surface Gravity Waves

5

Boundary conditions (i), (ii) & (iii) have to be linearized.

Assume small perturbations on a fluid at rest with constant mean depth *H*:

$$u = \delta u,\ \ \mathrm{v} = \delta \mathrm{v},\ \ w = \delta w,\ \ \underline{h = H + \delta h,}\ \ p = p_0(z) + \delta p,\ \ \rho = \rho_0(z) + \delta\rho, \cdots$$

Insert into (i), (ii) & (iii) and neglect products of perturbations.

non-linear boundary conditions:

linearized boundary conditions :

(i) $\boxed{w = 0 \quad \text{at} \quad z = 0}$ => (i) $\boxed{\delta w = 0 \quad \text{at} \quad z = 0}$

(ii) $\boxed{w = \frac{Dh}{Dt} \quad \text{at} \quad z = h(x,t)}$ => (ii) $\boxed{\delta w = \frac{\partial \delta h}{\partial t} \quad \text{at} \quad z = H + \delta h}$

(iii) $\boxed{\frac{Dp}{Dt} = 0 \quad \text{at} \quad z = h(x,t)}$ => (iii) $\boxed{\frac{\partial}{\partial t}\frac{\delta p}{\rho_0} = g\delta w \quad \text{at} \quad z = H + \delta h}$

Linearizing (iii) gives $\frac{\partial \delta p}{\partial t} + \delta w \frac{\partial p_0}{\partial z} = 0$

With hydrostatic balance for the basic state, i.e. $\frac{\partial p_0}{\partial z} = -g\rho_0$

Surface Gravity Waves 6

Assume wave form for the perturbations:

$$\delta h = \hat{h}\exp[i(kx+\sigma t)], \quad \delta w = \hat{w}(z)\cdot\exp[i(kx+\sigma t)], \quad \text{etc.}$$

=> (i) $\boxed{\hat{w} = 0 \quad \text{at} \quad z = 0}$

(ii) $\boxed{\hat{w} = i\sigma\hat{h} \quad \text{at} \quad z = H + \delta h}$

Condition (iii) can be shown (with the help of (29) and $B + g/c^2 = 1/H_0$) to be equivalent to

(iii) $$\boxed{\frac{d\hat{w}}{dz} - \left[\frac{n_1 - n_2}{H_0} + g\frac{k^2}{\sigma^2 - f^2}\right]\hat{w} = 0 \quad \text{at} \quad z = H + \delta h} \quad (58)$$

Now we filter out the acoustic waves:

Anelastic approximation is $n_2 = n_3 = 0$

Boundary condition (iii) (eq. (58)) contains $(n_1 - n_2)$!
Therefore, if $n_2=0$ we must set also $n_1=0$ in (iii) to avoid spurious solutions because of an inconsistent approximation!

Surface Gravity Waves 7

We have to solve now the set of equations given by:

Equation (57) + boundary conditions (i), (ii) & (iii)
with $n_2 = n_3 = 0$ in (57) and in (iii) (equation (58))
and with $n_1 = 0$ only in (iii)!

=>

$$\frac{d^2\hat{w}}{dz^2} - \frac{n_1}{H_0}\frac{d\hat{w}}{dz} - n_4\sigma\frac{k^2}{\sigma^2 - f^2}\hat{w} = 0 \qquad (57a)$$

(i) $\hat{w} = 0 \quad \text{at} \quad z = 0$

(ii) $\hat{w} = i\sigma\hat{h} \quad \text{at} \quad z = H + \delta h$

(iii) $\frac{d\hat{w}}{dz} - g\frac{k^2}{\sigma^2 - f^2}\hat{w} = 0 \quad \text{at} \quad z = H + \delta h$ (58a)

It is possible to solve above set for all wave numbers.
We study long and short waves independently.

Surface Gravity Waves 8

1.) Long waves

$kH \ll 1 \quad and \quad kH_0 \ll 1$, i.e. horizontal scale >> vertical scale

=> It is OK to make the hydrostatic approximation ($n_4 = 0$ in (57a)).

=> $$\frac{d^2\hat{w}}{dz^2} - \frac{n_1}{H_0}\frac{d\hat{w}}{dz} = 0$$ (57b)

Solution: (for $n_1=1$)

$$\delta w(x,z,t) \propto \sigma \underbrace{\frac{1-e^{\frac{z}{H_0}}}{e^{\frac{H}{H_0}}-1}}_{\doteq \hat{w}(z)} \exp\left[i\left(kx+\sigma t\right)\right] \quad (59)$$

$\doteq \hat{w}(z) \xrightarrow{z\to 0} 0$, i.e. $\delta w = 0$ at $z = 0$. (i) ok!

Verify by inserting $\hat{w}$ into (57b)!

From (ii) $\hat{w} = i\sigma\hat{h} \quad$ at $\quad z = H$ $\overset{i=\exp(i\frac{\pi}{2})}{=>}$ $$\delta h(x,t) \propto \exp\left[i\left(kx+\sigma t+\frac{\pi}{2}\right)\right]$$

Phase shift of 90^0 between δh and δw.

Surface Gravity Waves 9

Solution (59) has also to fulfill the boundary condition (iii) (equation (58a)).

(iii) $$\frac{d\hat{w}}{dz} - g\frac{k^2}{\sigma^2 - f^2}\hat{w} = 0 \quad \text{at} \quad z = H \qquad (58a)$$

Insert $\hat{w}(z) \propto \sigma \dfrac{1 - e^{\frac{z}{H_0}}}{e^{\frac{H}{H_0}} - 1}$ into (58a) and evaluate at z = *H*.

=> $$\sigma^2 = f^2 + gH_0k^2\left[1 - \exp\left(-\frac{H}{H_0}\right)\right] \qquad (59a)$$

dispersion relationship for <u>long</u> surface gravity waves

Surface Gravity Waves 10

Closer examination of (59a):

$$\sigma^2 = f^2 + gH_0k^2\left[1-\exp\left(-\frac{H}{H_0}\right)\right] \quad (59a)$$

Case $H << H_0$

(Shallow layer, much shallower than density scale height)

From (59a) with Taylor expansion of exp(-H/H_0) around H/H_0= 0:

$$1-\frac{H}{H_0}+O((H/H_0)^2)$$

=>

$$c_k = -\frac{\sigma}{k} \approx \pm\sqrt{\frac{f^2}{k^2}+gH}$$

Phase speed of long surface gravity waves in a shallow layer.

Examples:

* long waves on a boundary layer inversion (H ~ 1km) travel with c_k ~ 100m/s

* long waves on ocean surface (H ~ 4km) travel with c_k ~ 200m/s (tsunamis generated by underwater earthquakes)

Surface Gravity Waves 11

Case $H >> H_0$

(Deep layer, much deeper than density scale height)

$$\sigma^2 = f^2 + gH_0k^2\left[1 - \exp\left(-\frac{H}{H_0}\right)\right] \tag{59a}$$

From (59a) we obtain in this case for the phase speed

$$c_k = -\frac{\sigma}{k} = \pm\sqrt{\frac{f^2}{k^2} + gH_0}$$

Phase speed of long surface gravity waves in a deep layer.

These waves have a phase speed of about 260m/s for a density scale height H_0 of 7km.

Surface Gravity Waves 12

Remarks:

1.)

In a shallow layer
($H << H_0$)

$$c_k = \pm\sqrt{\frac{f^2}{k^2} + g\underline{H}}$$

In a deep layer
($H >> H_0$)

$$c_k = \pm\sqrt{\frac{f^2}{k^2} + g\underline{H_0}}$$

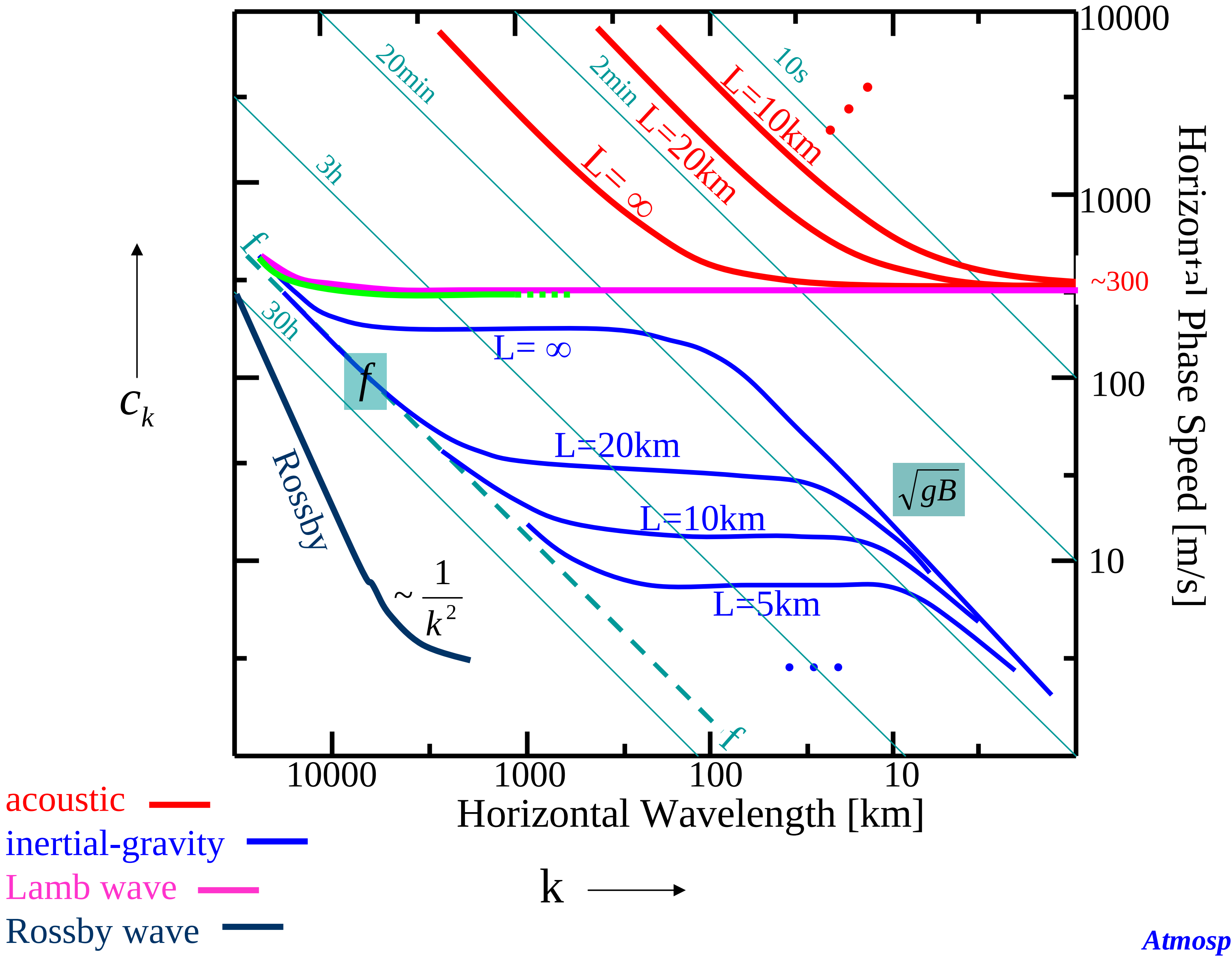
Always the smaller of the two depths!

2.) In an incompressible fluid ($n_1=0$) the phase speed is given by the expression for the shallow layer (because the density scale height $H_0 = \infty$). These long surface waves are often referred to as 'shallow water waves'. The corresponding 'shallow water equations' are extensively used for designing and testing of numerical schemes.

3.) Phase speed in a deep layer looks a bit like the phase speed of the Lamb wave

$c_k = \pm\sqrt{\frac{f^2}{k^2} + c^2}$ only $c^2 \longleftrightarrow gH_0$, but these terms are of similar magnitude!

Dispersion curve of long surface gravity waves in a deep layer —



Surface Gravity Waves 13

2.) Short waves

$kH >> 1 \quad and \quad kH_0 >> 1$, i.e. horizontal scale << vertical scale!

=> Hydrostatic approximation not OK!

Effect of rotation can be neglected!

Solution can be shown to be: (verify by inserting into (57a) and (58a)!)

$$\delta w \propto \underbrace{-\sigma \frac{\sinh(kz)}{\sin(kH)}}_{} \cdot \exp\left[n_1 \frac{z-H}{2H_0} \right] \cdot \exp[i(kx+\sigma t)] \quad (60)$$

$\xrightarrow{z \to 0} 0$, i.e. $\delta w = 0$ at $z = 0$. (i) ok!

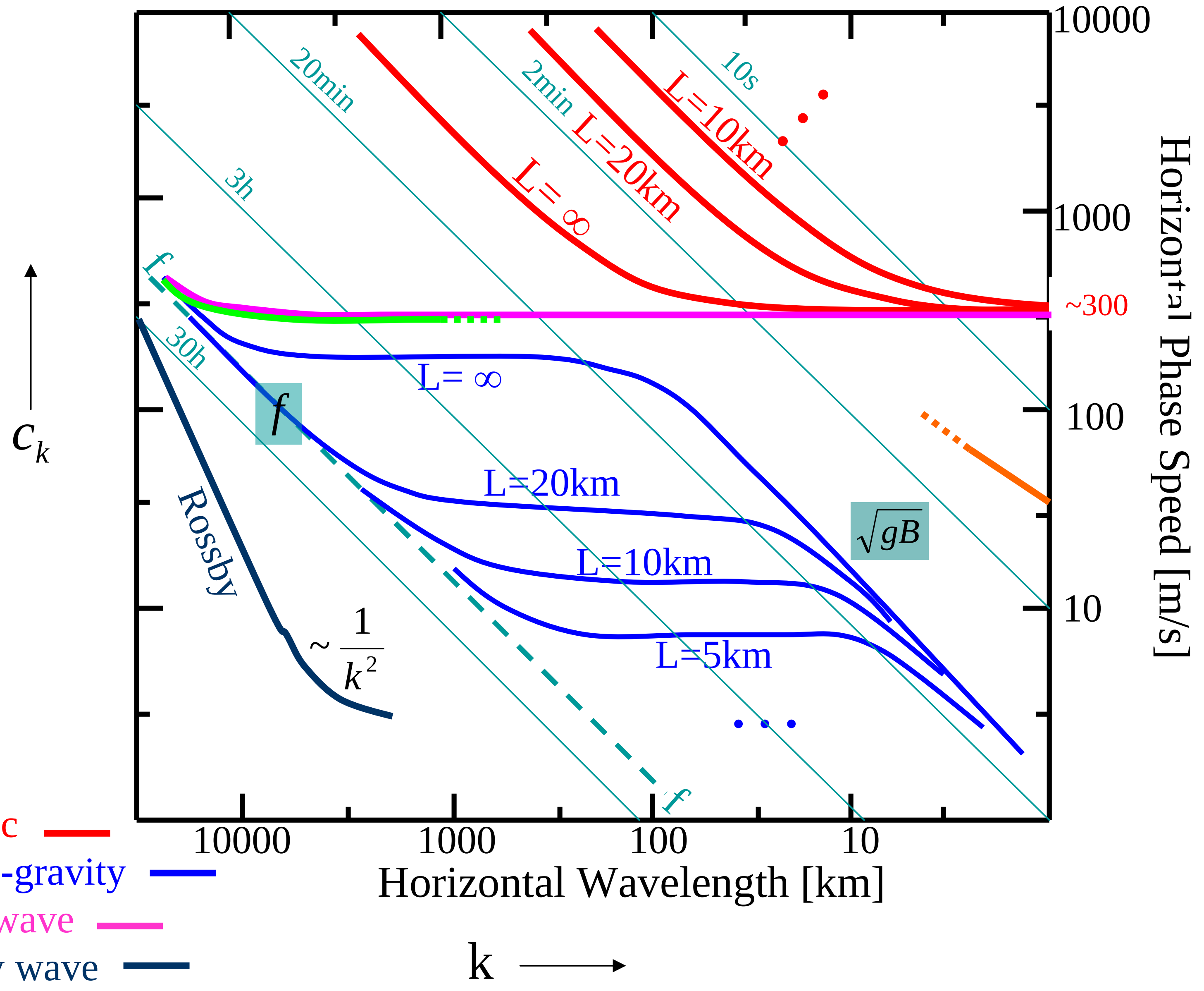
with the dispersion relationship $\sigma^2 = gk \tanh(kH)$

For very large *kH*: $\sigma^2 \approx gk \quad => \quad c_k \approx \pm\sqrt{\frac{g}{k}}$

$c_k \approx$ 40m/s for L_x=1km, i.e. much slower than long waves!

Example for this wave type: Ripples on a pond.

Dispersion curve of short surface gravity waves —



acoustic —

inertial-gravity —

Lamb wave —

Rossby wave —

long surface gravity waves in a deep layer —

Equatorial Waves

Why do we have to take an extra look at the equatorial region?

What is different at the equator from other latitudes?

Coriolis parameter is zero and changes sign. $\boxed{f = 0 \text{ but } \frac{\partial f}{\partial y} \neq 0\,!}$

Since we have not assumed anywhere $f \neq 0$ when we derived the wave solutions of the linearized basic equations the equatorial waves should be "contained" in these solutions (we just have to let $f \longrightarrow 0$), shouldn't they?

No, $f \longrightarrow 0$ in the solutions we have derived is not sufficient to find all equatorial waves!

Equatorial Waves 2

Why have we not found the special equatorial waves?

We neglected variations in the meridional direction ($\partial/\partial y \equiv 0$) when we studied inertial-gravity waves and acoustic waves.

No Rossby waves in this case because $\beta = 0$!

When we allowed variation with y to study Rossby waves we filtered out acoustic waves and inertial-gravity waves!

Equatorial Waves 3

Away from the equator this is ok!

Acoustic waves, inertial-gravity waves and Rossby waves are (nearly) independent solutions because the restoring mechanisms responsible for these 3 wave classes are well developed and distinct.

Near the equator this is no longer true (because the Coriolis force is weak & changes sign) and hybrid wave types can occur (mixed Rossby-gravity waves).

=> we have to study Rossby waves and inertial-gravity waves together!

Acoustic waves can be filtered out.

Equatorial Waves 4

To study equatorial waves we

- use the sallow water equations (i.e. 2d-problem (*x*,*y*)!)
- make the equatorial β-plane approximation: $f \approx f_0 + \beta y = \beta y$
- linearize again about a state at rest (basic state wind $\equiv 0$)

=>

$$\begin{cases} \dfrac{\partial \delta u}{\partial t} - \beta y \delta \mathrm{v} + g \dfrac{\partial \delta h}{\partial x} = 0 \\ \dfrac{\partial \delta \mathrm{v}}{\partial t} + \beta y \delta u + g \dfrac{\partial \delta h}{\partial y} = 0 \\ \dfrac{\partial \delta h}{\partial t} + H \left(\dfrac{\partial \delta u}{\partial x} + \dfrac{\partial \delta \mathrm{v}}{\partial y} \right) = 0 \end{cases} \tag{61}$$

Equatorial Waves

5

Assuming waves in *x*-direction for the perturbations:

$$(\ \delta u,\ \delta \mathrm{v},\ \delta h\ ) = (\ \hat{u}(y),\ \hat{\mathrm{v}}(y),\ \hat{h}(y)\ )\ \cdot \exp\left[i\left(kx + \sigma t \right) \right]$$

Inserting into system of equations (61) leads to the following 2-order ordinary differential equation for $\hat{\mathrm{v}}(y)$

$$\frac{d^2 \hat{\mathrm{v}}}{dy^2} + \left(\frac{\sigma^2}{gH} - k^2 + \frac{k\beta}{\sigma} - \frac{\beta^2 y^2}{gH} \right) \hat{\mathrm{v}} = 0 \qquad (62)$$

Change to non-dimensional forms of y, k and σ :

$$\underline{y}^2 = \frac{\sqrt{gH}}{\beta} \underline{\lambda}^2, \quad \underline{k}^2 = \frac{\beta}{\sqrt{gH}} \underline{\mu}^2, \quad \underline{\sigma}^2 = \sqrt{gH}\, \beta\, \underline{\omega}^2$$

Equatorial Waves 6

With these new variables (62) has the form

$$\frac{d^2\hat{v}}{d\lambda^2}+\left(\omega^2-\mu^2+\frac{\mu}{\omega}-\lambda^2\right)\hat{v}(\lambda)=0 \qquad (63)$$

Because the equatorial β-plane approximation is not valid beyond ±30^0 away from the equator we have to confine the solutions close to the equator if they are to be good approximations to the exact solutions.
=> boundary condition:

$$\hat{v} \rightarrow 0 \quad \text{for large } |\lambda| \qquad (63a)$$

Equatorial Waves 7

Find solutions for (63) + boundary condition (63a).

$$\frac{d^2\hat{v}}{d\lambda^2}+\Bigl(\underbrace{\omega^2-\mu^2+\frac{\mu}{\omega}}_{E}-\lambda^2\Bigr)\hat{v}(\lambda)=0 \qquad (63)$$

<u>Remark:</u> Equation (63) is of the same form as the <u>Schrödinger equation</u> for a quantum particle in a 1-dim. harmonic potential x^2:

$$\frac{d^2\Psi}{dx^2}+(\underline{E}-x^2)\Psi(x)=0$$

Solutions are possible only for discrete values of E (<u>discrete spectrum</u>):

$$E_n=2n+1 \quad \text{where} \quad n=0,1,2,\cdots$$

Solutions of (63)+boundary condition (63a) exist only for

$$\omega^2-\mu^2+\frac{\mu}{\omega}=2n+1 \quad \text{where} \quad n=0,1,2,\cdots \qquad (64)$$

dispersion relationship

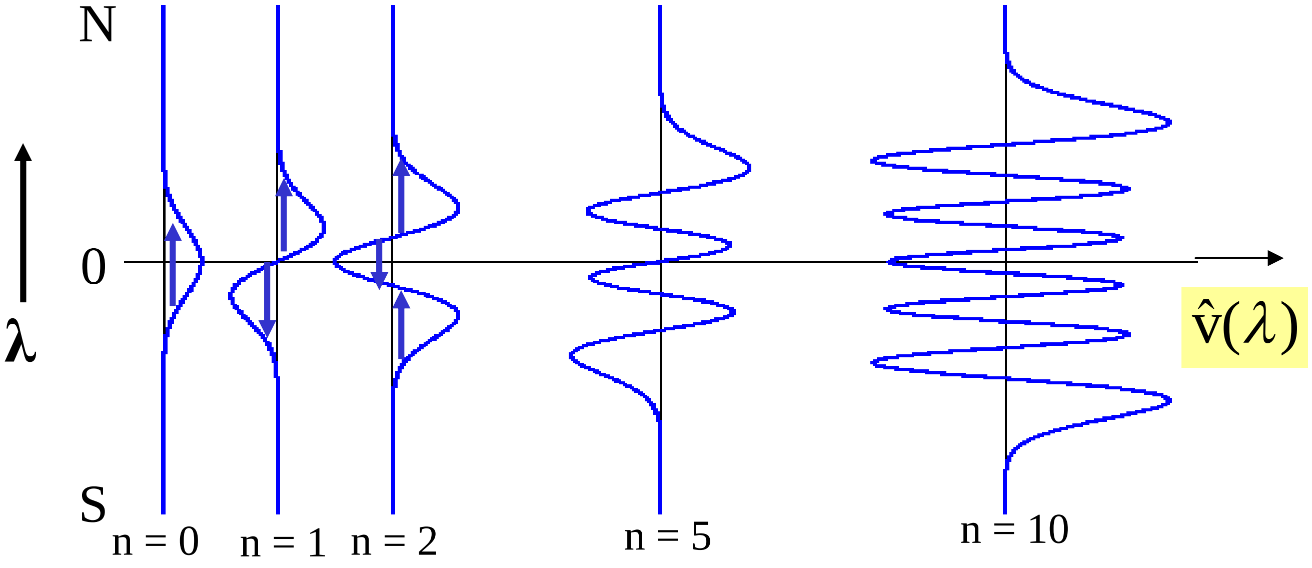
Equatorial Waves

Solutions for $\hat{\mathrm{v}}(\lambda)$ are given by

$$\hat{\mathrm{v}}(\lambda) = \mathrm{v}_0 \exp\left(-\frac{\lambda^2}{2}\right) \cdot H_n(\lambda) \quad \text{where} \;\; n = 0, 1, 2, \cdots$$

Here H_n is the Hermite polynomial of order n.

$$H_0 = 1 \qquad H_1 = 2\lambda \qquad H_2 = 2\left(2\lambda^2 - 1\right) \qquad \cdots$$



Exercise: Insert the solution for n=1 into (63) and verify that it indeed fulfills this equation and the boundary condition (63a) only if (64) for n=1 is satisfied.

Equatorial Waves

9

Inspection of the dispersion relationship (64):

(64) <=> $\boxed{\omega^3 - (\mu^2 + 2n + 1)\omega + \mu = 0}$ (65)

Cubic equation in the frequency ω. We expect 3 distinct roots.

Find roots first for the case that $\boxed{n \neq 0}$

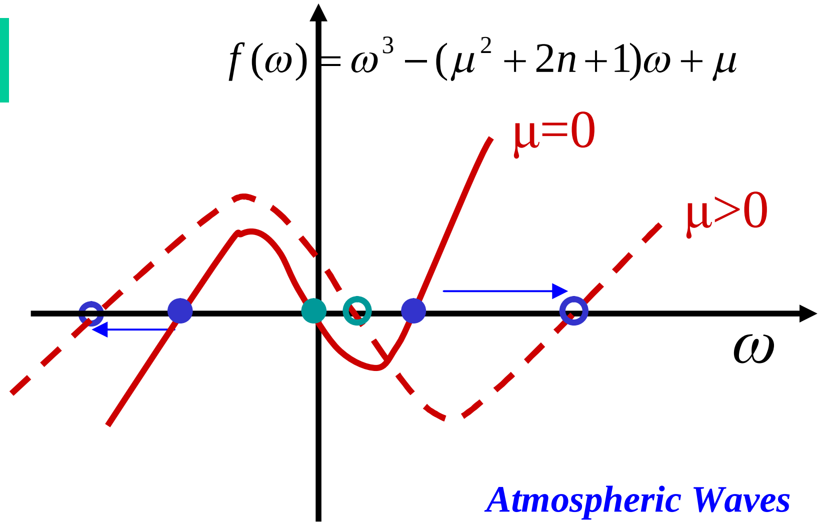
For $\boxed{\mu = 0}$ => $\boxed{\omega_{1,2} = \pm\sqrt{2n+1}}$ & $\boxed{\omega_3 = 0}$

For $\boxed{\mu \neq 0}$ we can find good approximations to the 3 roots by considering the cases

$\boxed{\omega^2 \sim \mu^2}$ & $\boxed{|\omega| << \mu}$

Why?

To see why



Equatorial Waves

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$$\omega^3 - (\mu^2 + 2n + 1)\omega + \mu = 0 \qquad (65)$$

$$\omega^2 \sim \mu^2 \quad \Rightarrow \quad \mu \ll |\omega^3|, \; \mu \ll \left|\left(\mu^2 + 2n + 1\right)\omega\right| \qquad \left(\text{if } \; \omega^2 > 1 \, , \; \mu^2 > 1\right)$$

=> neglect μ in (65) =>

$$\omega_{1,2} = \pm\sqrt{\mu^2 + 2n + 1} \qquad (66)$$

$$|\,\omega\,| \ll \mu \quad \Rightarrow \quad \omega^3 \ll \left(\mu^2 + 2n + 1\right)\omega + \mu$$

=> neglect ω^3 in (65) =>

$$\omega_3 = \frac{\mu}{\mu^2 + 2n + 1} \qquad (67)$$

Back to dimensional variables k and σ:

(66) <=>

$$\sigma_{1,2} = \pm\underline{\sqrt{gH}\,k} \cdot \sqrt{1 + \frac{\beta(2n+1)}{k^2\sqrt{gH}}}$$

pair of gravity waves

(one pair for each **n** = 1, 2, 3, …)

(67) <=>

$$\sigma_3 = \frac{\underline{\beta k}}{\underline{k^2} + \dfrac{\beta(2n+1)}{\sqrt{gH}}}$$

westward propagating Rossby wave (one for each **n** = 1, 2, 3, …)

Equatorial Waves

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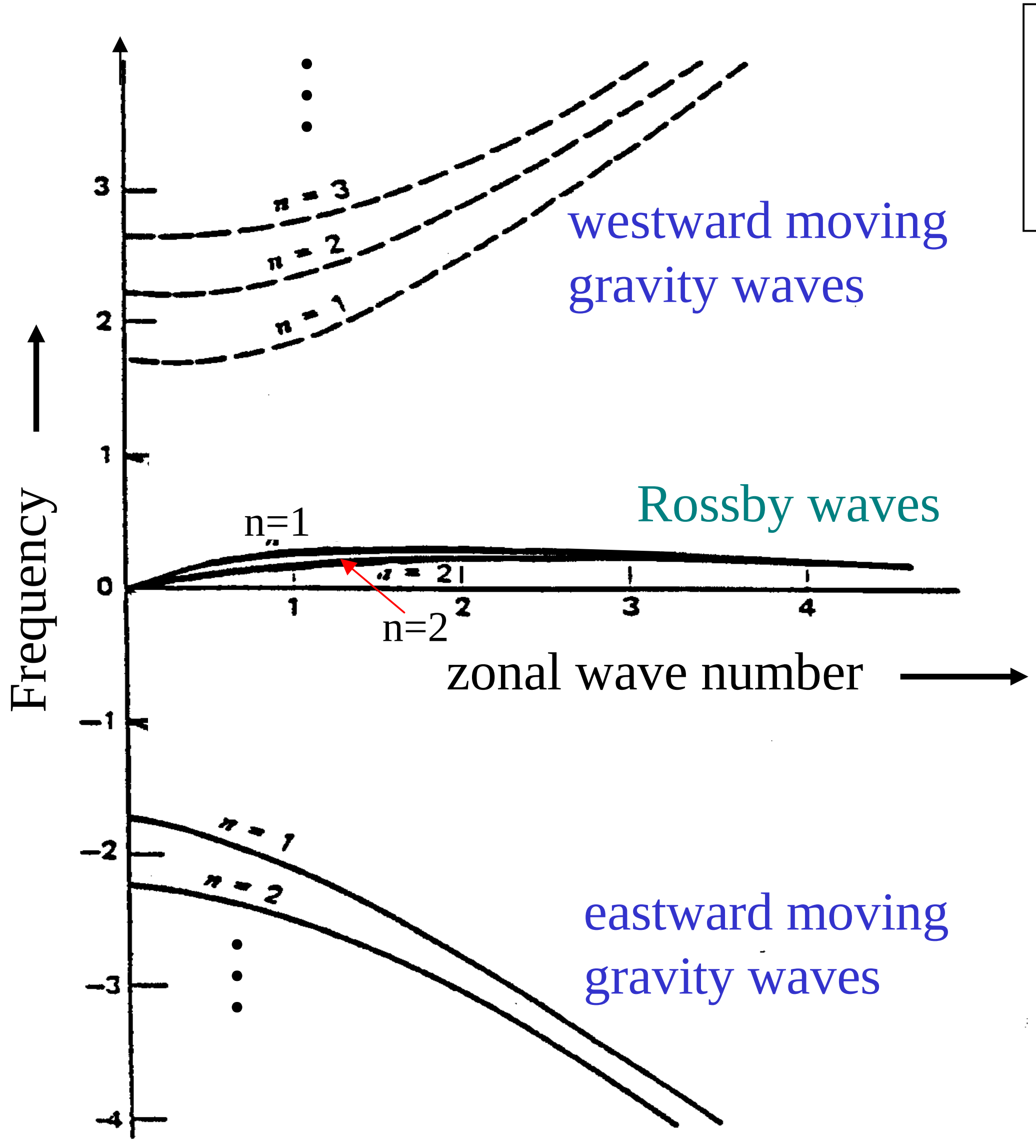

From T Matsuno (1966)
“Quasi-geostrophic Motions
in the Equatorial Area”,
Journal Met. Soc. of Japan.

Equatorial Waves 12

<u>Case $n = 0$:</u>

$$\omega^3 - (\mu^2 + 2n + 1)\omega + \mu = 0 \quad (65)$$

Dispersion relationship (65) can be factorized

$$(\omega - \mu)(\omega^2 + \omega\mu - 1) = 0$$

Root 1: $\omega = \mu$ This root is not acceptable because a division by ω - μ is required in deriving (62) from (61)!

Root 2: $\omega_1 = -\frac{\mu}{2} - \sqrt{\frac{\mu^2}{4} + 1}$ <=> $\sigma_1 = -\frac{\sqrt{gH}\,k}{2}\left(1 + \sqrt{1 + \frac{4\beta}{\sqrt{gH}\,k^2}}\right)$ (68)

Root 3: $\omega_2 = -\frac{\mu}{2} + \sqrt{\frac{\mu^2}{4} + 1}$ <=> $\sigma_2 = -\frac{\sqrt{gH}\,k}{2}\left(1 - \sqrt{1 + \frac{4\beta}{\sqrt{gH}\,k^2}}\right)$ (69)

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(68) $$\sigma_1 = -\frac{\sqrt{gH}\,k}{2}\left(1+\sqrt{1+\frac{4\beta}{\sqrt{gH}\,k^2}}\right)$$

$\beta = 0 \;\Rightarrow\; \sigma_1 = -\sqrt{gH}\,k$
= frequency of shallow water gravity waves,
i.e. this is a gravity wave

For <u>large k</u> $\quad \sigma_1 \to -\sqrt{gH}\,k$

For $k \to 0$ $\quad \sigma_1 \to -\left(\sqrt{gH}\,\beta\right)^{1/2}$

eastward moving gravity waves

(69) $$\sigma_2 = -\frac{\sqrt{gH}\,k}{2}\left(1-\sqrt{1+\frac{4\beta}{\sqrt{gH}\,k^2}}\right)$$

This is some kind of a Rossby-type wave
because for $\beta = 0 \;\Rightarrow\; \sigma_2 = 0$

For <u>large k</u> $\quad \sigma_2 \to \dfrac{\beta}{k}$

(as for Rossby waves)

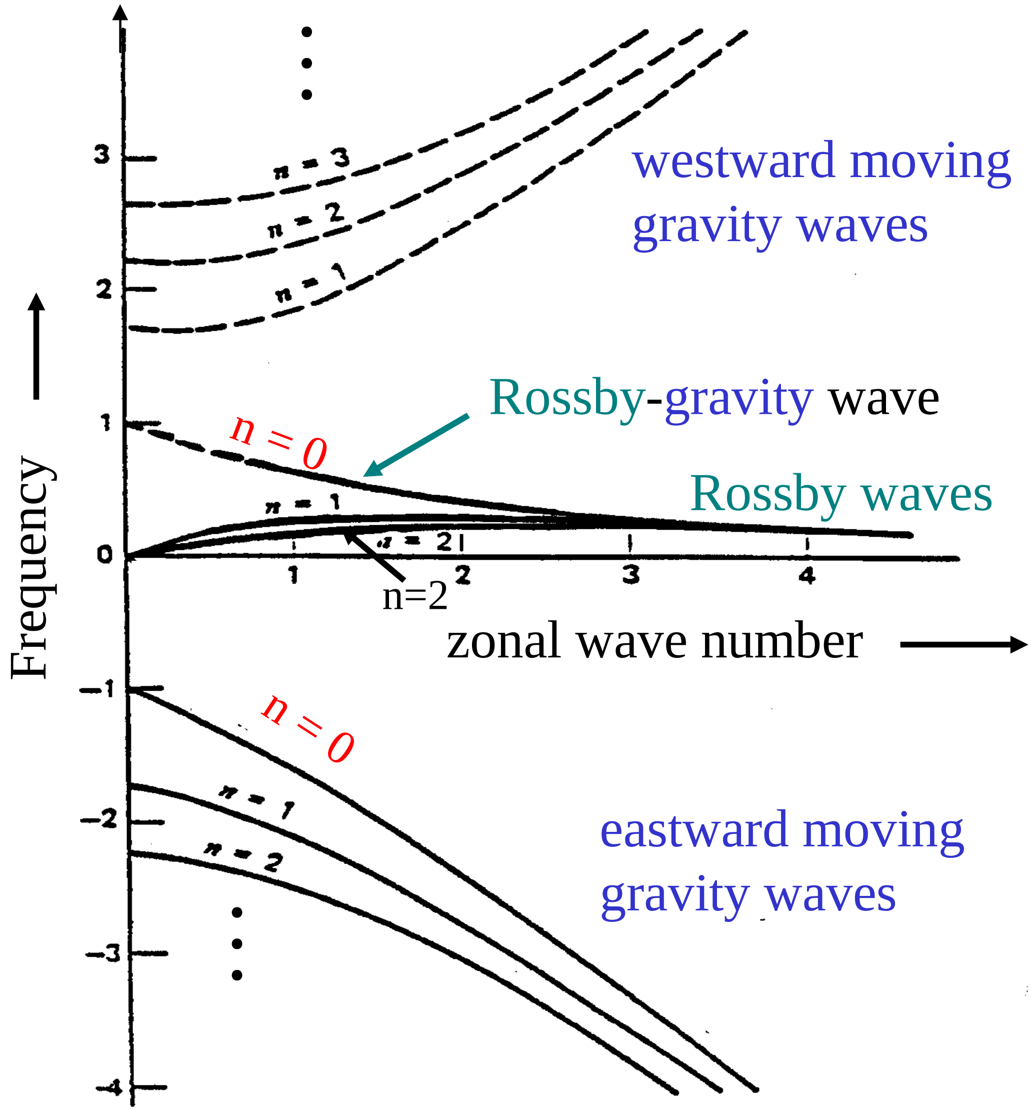
For $k \to 0$ $\quad \sigma_2 \to +\left(\sqrt{gH}\,\beta\right)^{1/2}$

(as for gravity waves)

=> called mixed Rossby-gravity waves
(westward moving)

Equatorial Waves

From T Matsuno (1966) “Quasi-geostrophic Motions in the Equatorial Area”, Journal Met. Soc. of Japan, **44**, 25-42.



Equatorial Waves

Structure of mixed Rossby-gravity waves:

n=0

=>
$$\delta \mathrm{v}(x,y,t) = \mathrm{v}_0 \exp\left(-\frac{\beta y^2}{2\sqrt{gH}}\right) \exp\left[i\left(kx+\sigma t\right)\right]$$

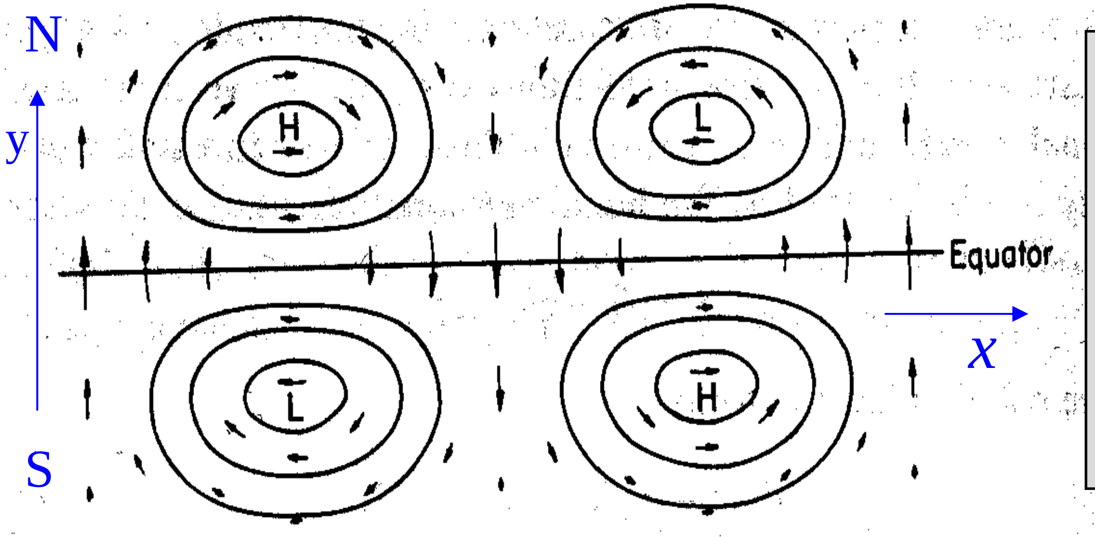
From (61) we obtain:

$$\delta u(x,y,t) = \underline{i}\, A_u(\sigma,k)\cdot \underline{y}\cdot \delta \mathrm{v}(x,y,t)$$

$$\delta h(x,y,t) = \underline{i}\, A_h(\sigma,k)\cdot \underline{y}\cdot \delta \mathrm{v}(x,y,t)$$

$$i = \exp\left(i\frac{\pi}{2}\right)$$

Phase shift of 90^0
between δ*u* and δv
& between δ*h* and δv!



Plan view of horizontal velocity and height perturbation associated with an equatorial <u>Rossby-gravity wave</u>. (Adapted from Matsuno,1966: Quasi-geostrophic motions in the equatorial area. *J. Meteo. Soc. Japan*, **44**, 25-42)

Equatorial Waves

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Equatorial Kelvin Waves

= Waves with zero meridional velocity everywhere, i.e. $\delta v \equiv 0$

(Reminder: Lamb waves $\delta w \equiv 0$)

In this case equation (63) is redundant!

Derive solutions from set of linearized shallow water equations (61) with $\delta v \equiv 0$ & boundary condition (solution must be confined close to the equator).

Solution:

$$\delta u(x,y,t) = u_0 \exp\left(-\frac{\beta k}{2\sigma} y^2\right) \exp\left[i(kx - \sigma t)\right] \quad (70)$$

with dispersion relationship

$$\sigma = \pm\sqrt{gH}\,k \quad (71)$$

Only " + " in (71) is valid solution, " - " violates the β-plane approximation (since δu in (70) is growing not decaying with y in this case!)

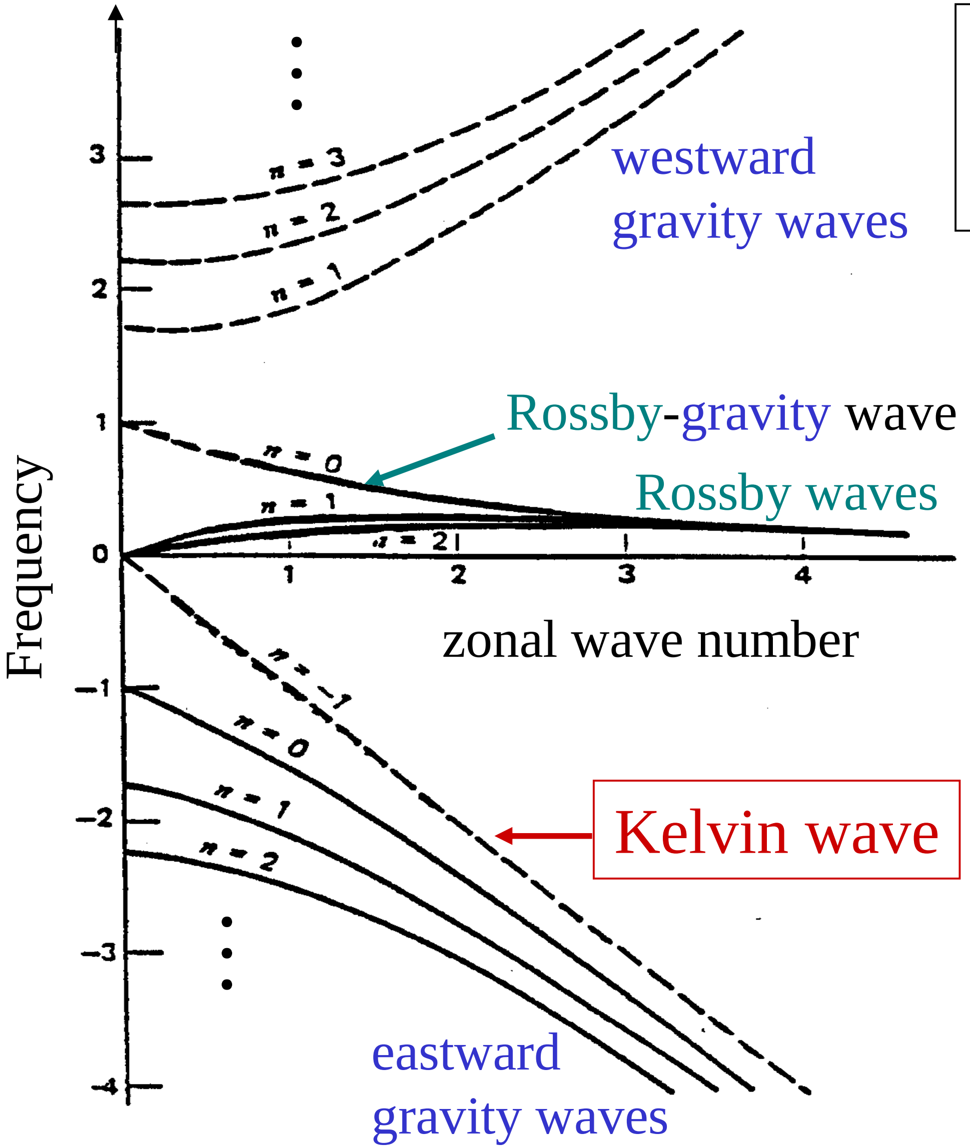
=> $c_k = +\sqrt{gH}$

phase speed of shallow water gravity waves!

Kelvin waves move only eastward! (non-dispersive)

=> They are a form of gravity waves.

Equatorial Waves



From T Matsuno (1966)
“Quasi-geostrophic Motions
in the Equatorial Area”,
Journal Met. Soc. of Japan.

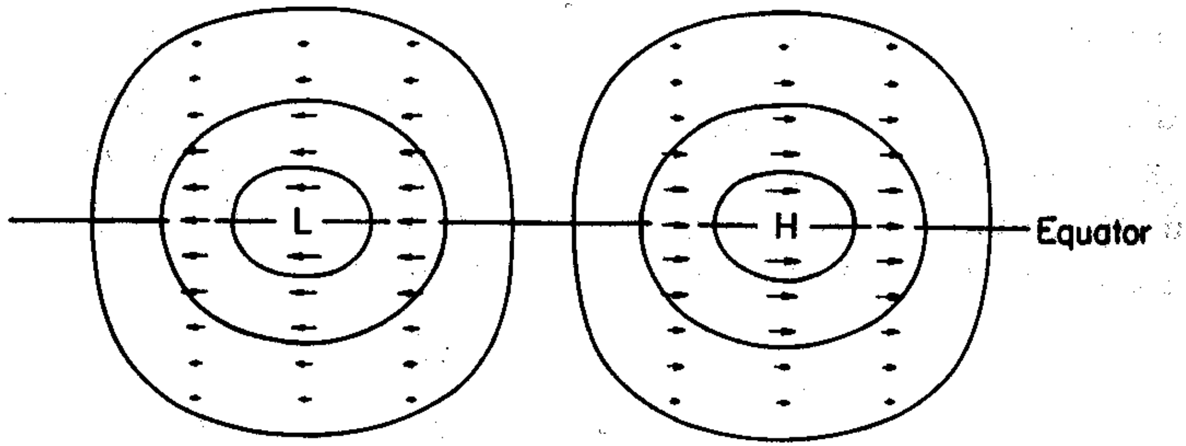
Equatorial Waves

Structure of equatorial Kelvin waves:

$$\delta u(x,y,t) = u_0 \exp\left(-\frac{\beta k}{2\sigma} y^2\right) \exp\left[i\left(kx - \sigma t\right)\right] \tag{70}$$

$$\delta v \equiv 0$$

$$\delta h(x,y,t) = A(\sigma,k)\,\delta u(x,y,t)$$



Plan view of horizontal velocity and height perturbations associated with an equatorial Kelvin wave.
(Adapted from Matsuno, T. 1966:Quasi-geostrophic motions in the equatorial area. *J. Meteo. Soc. Japan*,**44**,25-42)

Meridional force balance is an exact geostrophic balance between u and the meridional pressure gradient.
Existence of Kelvin waves is thanks to the change in sign of Coriolis parameter at equator!
Zonal force balance is that of an eastward moving shallow water gravity wave.

Ocean Kelvin waves along coastlines are more common than atmospheric Kelvin waves.

Equatorial Waves

Characteristics of the dominant observed Kelvin and Rossby-gavity waves of planetary scale in the atmosphere:

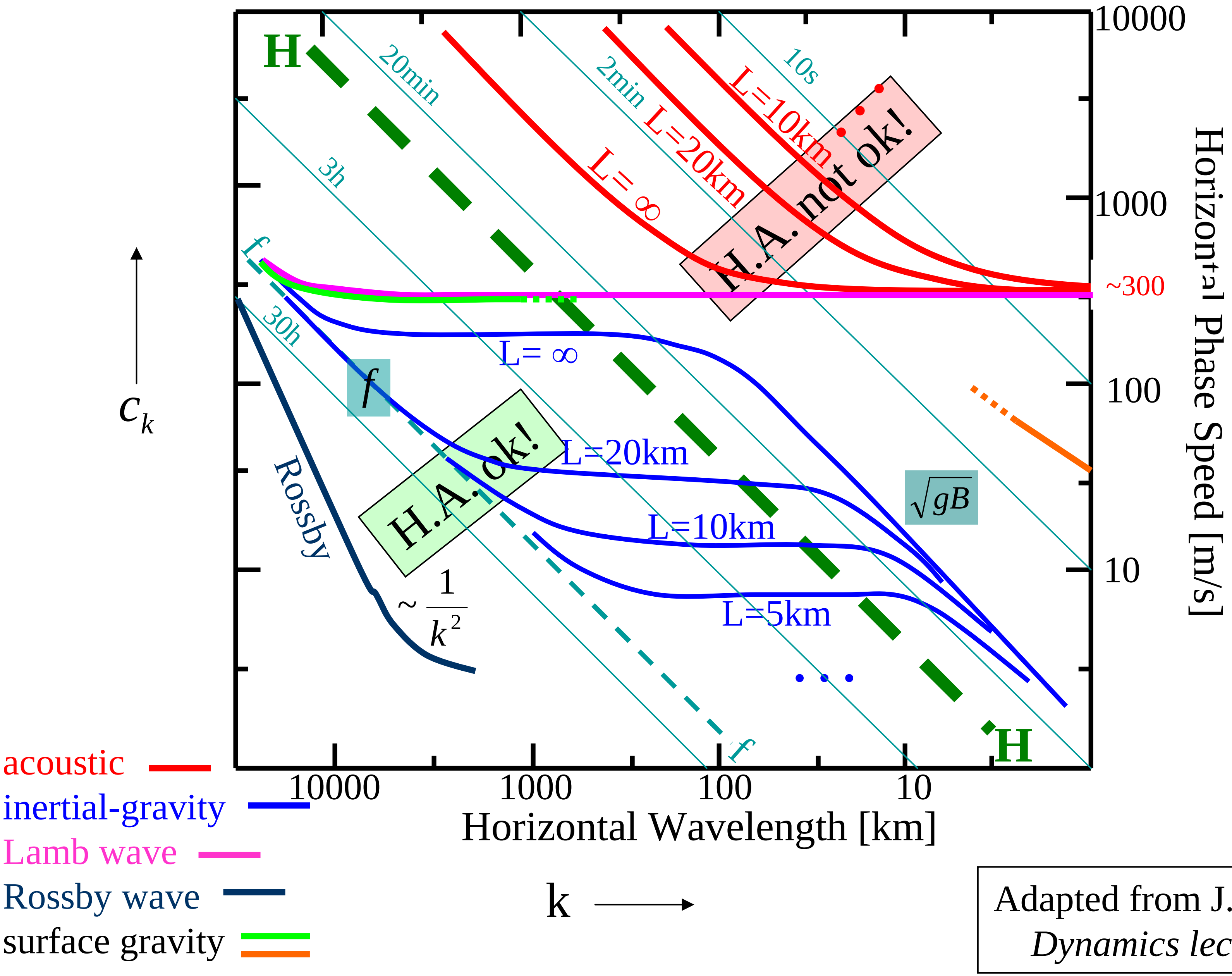
	Kelvin waves	Rossby-gravity waves
Period	15 days	4-5 days
Zonal wave number	1-2	4
Vertical wavelength	6-10 km	4-8 km
Average phase speed relative to the ground	+25 m/s	-23 m/s
Observed when mean flow is	easterly	westerly
Discovered by	Wallace & Kousky (1968)	Yanai & Maruyama (1966)

Form J. R. Holton: An Introduction to Dynamic Meteorology

These waves play an important role in the generation of the quasi-biennial oscillation (QBO) in the zonal wind of the equatorial stratosphere.

Thank you very much for your attention.

SUMMARY OF WAVES IN A COMPRESSIBLE ATMOSPHERE



Adapted from J. S. A. Green:
Dynamics lecture notes

Introduce first some simplifications:

Change of variable: Replace T by $\boxed{\Theta \equiv \ln\theta}$ where

$$\boxed{\theta \equiv T\left(\frac{p_0}{p}\right)^{\kappa}}$$ (potential temperature)

Simplification: Neglect variation in y $\boxed{\frac{\partial}{\partial y} \equiv 0}$ $\left(e.g. \frac{\partial f}{\partial y} = 0\right)$

Coordinates are now only (x, z, t)!

Question: Has this simplification serious consequences for the wave solutions?

Answer: Yes! The Rossby wave solution has been suppressed!! Rossby waves can only form if the Coriolis parameter f changes with latitude. (Detailed discussion of Rossby waves will follow later in this course.)

Dependent variables (unknowns) are now u, v, w, ρ, p, Θ and they are functions only of x, z and t.

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