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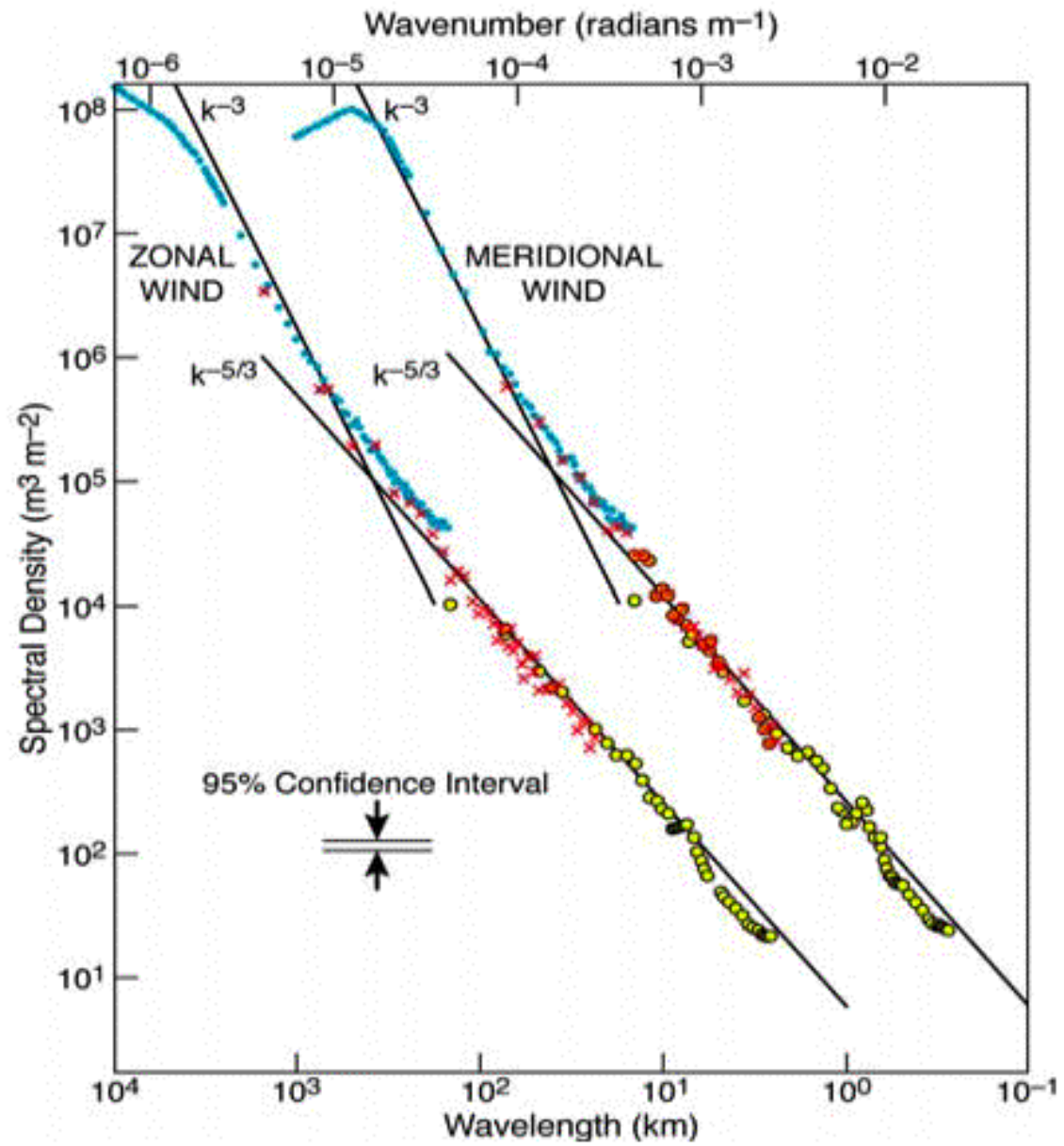
# **Stochastic Sub-Grid Scale Models and the Spectral Kink**

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1. Error growth versus mean energy
2. Kinetic energy spectra: the kink
3. Error growth rates revisited
4. Rotational and divergent norms
5. Proposed experiments
6. KE spectra from spectral element GCM

- SGS stochastic models viewed as perturbations
- How do these feed-back to larger scales
- Error growth or difference spectra versus:
- Kinetic energy cascade



# Kinetic Energy Spectra: The Kink

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Gage and Nastrom (1986):  $k^{-3}$  to  $k^{-5/3}$  kink at  $\mathcal{O}(1000)$  km

Alternative theories:

Gage (1979), Lilly (1983): Upscale energy cascade

Tacit assumption: Error growth follows similar mechanism

Dewan (1979), Van Zandt (1982),

Waite and Bartello (2004):

Downscale energy cascade

Internal gravity waves not possessing PV – but not necessarily with high linear frequencies

# Error Growth Rates

Model truncation wave number  $k_N$ .

$E(k)$  the kinetic energy per unit wave number.

Lorenz (1969) and Lilly (1972):

Time for error at  $2k$  to propagate one octave to wave number  $k$  proportional to eddy turnover time

$$\tau(k) = k^{-3/2} E(k)^{-1/2}$$

Time  $\Omega(k_N)$  to propagate  $N_0$  octaves from  $k_N$  to  $k_L$

$$\Omega(k_N) = \sum_{n=0}^{N_0-1} \tau(2^n k_L)$$

# Error Growth Rates

2-D enstrophy-cascade turbulence:  $E(k) \sim k^{-3}$ ,  $\tau$  is independent of  $k$  and  $\Omega(k_N) \sim N_0$ , diverges as  $k_n \rightarrow \infty$   
3-D energy-cascade:  $E(k) \sim k^{-5/3}$ , so  $\tau \sim k^{-2/3}$

$$\Omega(k_N) \sim \tau(k_L), \quad k_N \rightarrow \infty$$

Implies for  $k^{-5/3}$  flow that neglecting the effect of unresolved scale variance on the (mean and variance of the) large scale flow can not be made arbitrarily small.

# Tangent Linear Model

Measure perturbation growth due to perturbation  $f(t)$ :

$$y(t) = L(t_0, t)y(t_0) + \int_{t_0}^t L(\tau, t)f(\tau)d\tau$$

If  $f$  is constant wrt time, largest impact on flow at time  $t$  is given by the dominant singular vectors of

$$\mathcal{L} = \int_{t_0}^t L(\tau, t)d\tau$$

# Error Growth Rates

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Palmer (2001):

$\Omega^{-1}$  is related to the dominant singular value of  $\mathcal{L}$

Norm on the right singular-vector space involves projection onto  $k_N$

Norm on the left singular-vector space maximizes growth onto  $k_N$

# Error Growth Revisited

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Tribbia and Baumhefner (2004):

- More akin to exponential growth of baroclinic disturbances with spectral peak in synoptic scales (wavenumbers 10-20) than an inverse cascade
- Initial errors cascade to seed errors into baroclinically active wavenumbers
- Similar to singular vector growth in ECMWF-EFS Palmer et al (1994)
- Threshold physics effect rapid impact of errors from unresolved scales

# Stochastic SGS Models

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- Stochastic SGS models generate higher-order moments and introduce variance at the unresolved scales
- Nonlinear feedback to the resolved scales of motion
- Inverse cascade process ?
- How do we measure/assess the impact on the resolved dynamics ?
  - Singular vectors of TLM
  - Rotational and divergent norms

# Rotational and Divergent Norms

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How do stochastic SGS influence mesoscale (divergent) and large scale (rotational) modes ?

Erhendorfer and Errico (1995):

Total energy (TE) and rotational norm  $R$

Need norm  $D$  for projection onto divergent modes

Del Sole (2004) : Growth of optimal perturbations

Maximize instantaneous energy growth rates in turbulent flows, short optimization times.

Koshyk, Hamilton and Mahlman (1999): Evidence for kink  
GFDL Skyhi model. 30 km resolution.

Spectra do not exhibit energy drain (tail up).

Bartello (2002): Explicitly resolve gravity waves

Semi-implicit or split-explicit time-stepping modifies gravity  
wave spectrum

Could eliminate or change the location of the kink

# Proposed Experiments

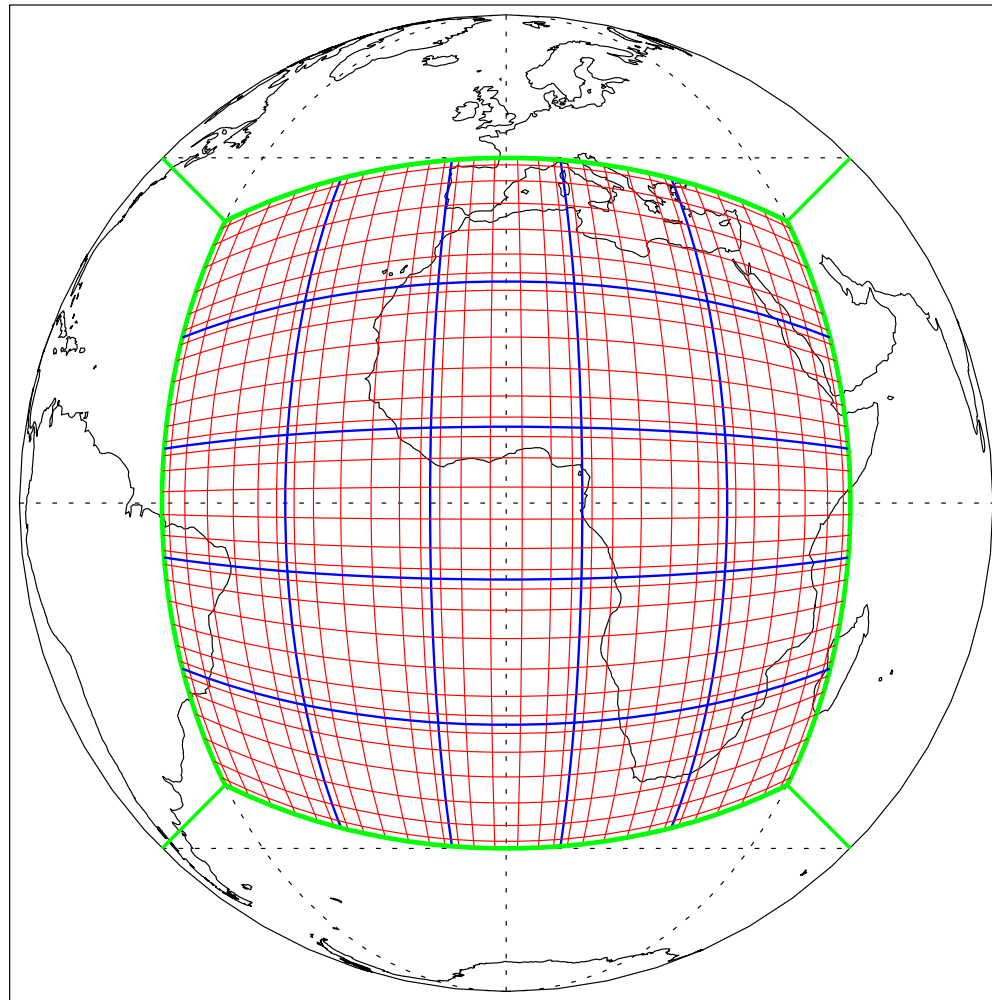
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- Problem with enough complexity to produce the spectral kink
  - forced baroclinic life-cycle
  - Thorncroft et al (1993), Polvani et al (2004)
- Examine divergence in phase space to understand how stochastic approaches introduce uncertainty in the resolved flow
- Gather evidence for producing the spectral kink and how the resolved-scale uncertainty propagates up-scale.

Skamarock (2004) reports spectral kink

- Non-hydrostatic NCAR WRF simulations
- $\mathcal{O}(5)$  km resolution
- hemispheric
- statistics based on shorter time scales
- resolve convection in mid-latitudes

# Cubed-Sphere



Sphere:  $\mathbf{v} = (v_1, v_2)$

$$u_i = G_{ij}u^j, \quad u^i = G^{ij}u_j, \quad G = \det(G_{ij})$$

The metric tensor for all six faces of the cube is

$$G_{ij} = \frac{1}{r^4 \cos^2 x^1 \cos^2 x^2} \begin{bmatrix} 1 + \tan^2 x^1 & -\tan x^1 \tan x^2 \\ -\tan x^1 \tan x^2 & 1 + \tan^2 x^2 \end{bmatrix}$$

$$r = (1 + \tan^2 x^1 + \tan^2 x^2)^{1/2}$$

Jacobi matrix

$$D = \begin{bmatrix} \cos \theta \partial \lambda / \partial x_1 & \cos \theta \partial \lambda / \partial x_2 \\ \partial \theta / \partial x_1 & \partial \theta / \partial x_2 \end{bmatrix}$$

$D^T D = G_{ij}$  and the cube to sphere mappings are

$$D^T \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad D \begin{bmatrix} u^1 \\ u^2 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

# High-Order Finite Elements

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- High-order spectral elements combine
  - Finite Element Method
  - Pseudospectral Method
- Continuous and discontinuous Galerkin
- CG enforces continuity at element boundaries
- DG analogous to conservative finite volume

# High-Order Finite Elements

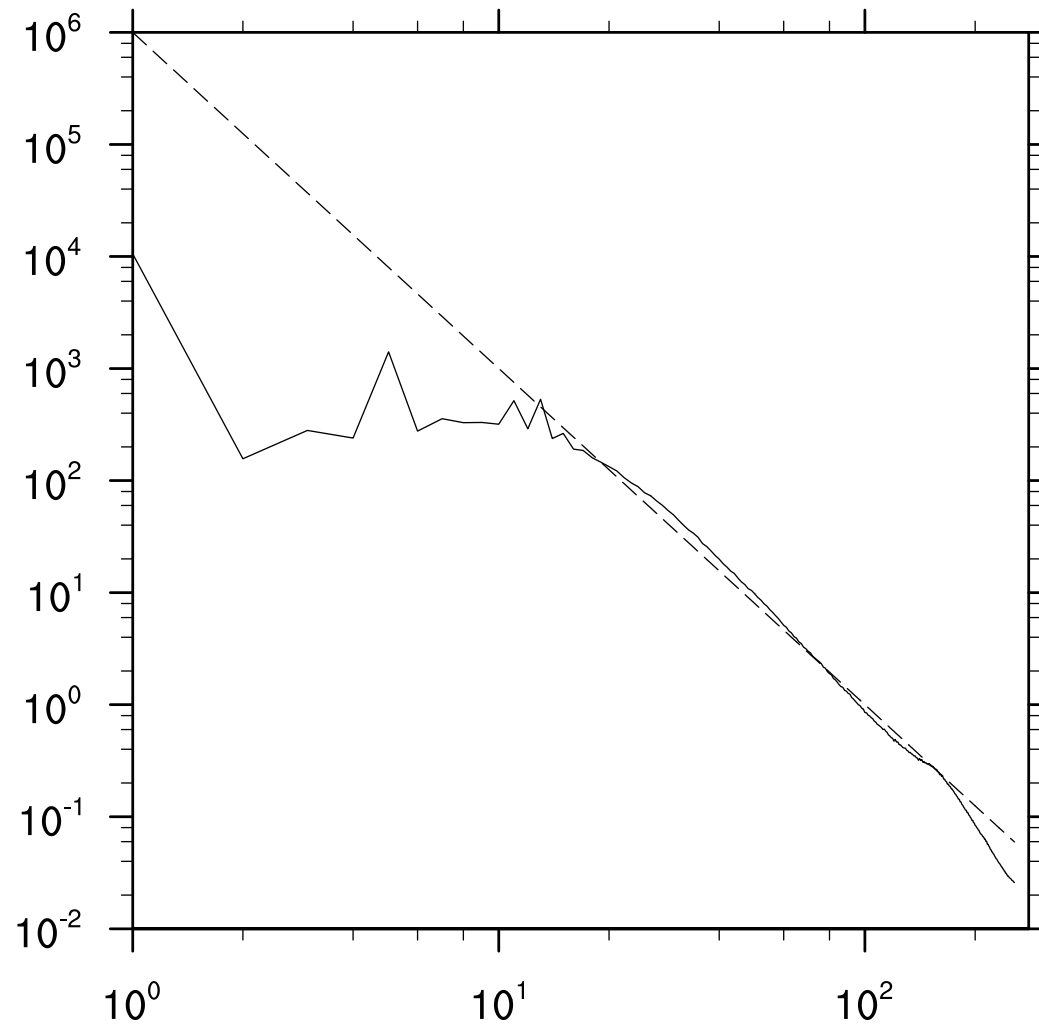
Lagrangian interpolants  $h_i$

$$\mathbf{v}_h^k(r_1, r_2) = \sum_{i=0}^N \sum_{j=0}^N \mathbf{v}_{ij} h_i(r_1) h_j(r_2)$$

Gauss-Legendre-Lobatto quadrature

$$\langle f, g \rangle_{GL} = \sum_{k=1}^K \sum_{i=0}^N \sum_{j=0}^N f^k(\xi_i, \xi_j) g^k(\xi_i, \xi_j) \rho_i \rho_j$$

KE Spectrum for HS Test Case Temporal Avg (Surface)



- Error growth different from KE cascade
- Upscale cascade of stochastic perturb to be studied
- Propose new tools to assess stochastic SGS models
- First reproduce spectral kink with GCM
- Projection on rotational  $R$  and divergent  $D$  modes
- Lindborg (1999) structure functions an alternative
- Speculate longer predictability time than eddy turn-over time in the  $-5/3$  regime